Science World Journal Vol. 20(No 1) 2025 https://dx.doi.org/10.4314/swj.v20i1.43
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

ANALYSING THE EXISTENCE AND UNIQUENESS SOLUTION OF A WILDFIRE MODEL WITH DIFFUSION AND CONVECTION OF MOISTURE

*[1, 2]Zhiri A.B.; [2]Ajala O.A.; [2]Peter A.; [2]Akindele A.O.

[1]Department of Mathematics, Federal University of Technology, Minna, Nigeria
[2]Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, Ogbomoso, Nigeria

*Corresponding Author Email Address: a.zhiri@futminna.edu.ng

ABSTRACT:
Wildfire spread modeling is governed by a complex system of non-linear partial differential equations (PDEs) that capture the intricate dynamics of wildfire behavior, including heat transfer and moisture interaction. A comprehensive understanding of these dynamics is critical for developing effective management, mitigation, and intervention strategies. In this study, temperature-dependent diffusion and convection terms are incorporated into the volume fraction of moisture, enriching the model framework and improving its accuracy in representing wildfire spread. To ensure the mathematical robustness of the model, the non-linear PDE system is transformed into a dimensionless form using appropriate dimensionless variables, facilitating the analysis of the equations. The model equations describe the dynamics of combustible forest material (CFM) in terms of the volume fractions of dry organic matter, moisture, coke, heat, and oxygen. The conditions for the existence and uniqueness of solutions to the model equations are rigorously established using the Lipschitz continuity criterion. The results confirm that unique solutions exist when the Lipschitz conditions are satisfied.

Keywords: Combustible forest material, Existence and uniqueness, Moisture diffusion and convection, Dimensionless variables, Lipschitz continuity

INTRODUCTION

Wildfires represent an escalating global threat to ecosystems, human settlements, and economies worldwide. This danger has become increasingly pronounced in recent years, exacerbated by climate change impacts. With global temperatures rising and drought conditions extending, both the frequency and severity of wildfires are projected to increase significantly in coming decades (Senande-Rivera *et al.,* 2022). These pressing concerns underscore the critical importance of developing accurate mathematical models and analytical tools to predict wildfire behaviour and mitigate their destructive impacts.

While wildfire ignition is inevitable occurring through natural causes such as lightning strikes or intense solar heat, as well as human-induced factors, that even a minimal spark can trigger a devastating inferno. Once initiated, wildfires can propagate at alarming rates, consuming forests at speeds reaching 23 km per hour and leaving widespread devastation in their wake (Kahanji *et al.,* 2019; Mangiameli *et al.,* 2021).

The growing scientific focus on wildfires stems from their increasingly catastrophic consequences, which are further amplified by climate change—a significant factor influencing both ignition likelihood and spread dynamics. Recent research has made considerable progress in developing sophisticated mathematical approaches to model these complex phenomena. Morgan (2024) explored a nonlinear reaction-diffusion system for wildfire propagation modelling, establishing the global-in-time existence and uniqueness of bounded mild solutions to the Cauchy problem under bounded initial conditions. Their analysis concluded that the model does not permit thermal blow-up scenarios. In parallel, Mitra et al. (2024) investigated wildfire spread through an advection–diffusion–reaction model incorporating both convective and radiative heat loss mechanisms. Their study analysed traveling wave (TW) existence in a one-dimensional wildfire spread model, employing both PDE solvers and shooting algorithms. Their results demonstrated strong alignment between theoretical predictions and numerical simulations, revealing critical dependencies of fire fronts on various model parameters.

Building on these approaches, Feckan and Pacuta (2018) developed a wildfire spread model utilizing Hamilton-Jacobi theory to demonstrate the existence of a classical solution to equation (1):

$$x_t = f_1\left(x_\phi, y_\phi\right),\ \ y_t = f_2\left(x_\phi, y_\phi\right), \quad \text{for} \quad \left(\phi, t\right) \in R \times \left(0, T\right), \tag{1}$$

where x_t, y_t, x_ϕ and y_ϕ denote partial derivatives with respect to t and ϕ respectively, of function $x\left(\phi, t\right)$ and $y\left(\phi, t\right)$

They established the existence of classical solutions under specific conditions and applied the method of characteristics to derive solutions in explicit form.

Wildfire dynamics fundamentally encompass two interconnected processes: (i) ignition and (ii) propagation (Harrison *et al.,* 2021). Despite extensive research efforts, these processes remain only partially understood (Crompton *et al.,* 2022). Mathematical and computational models offer a valuable framework for unravelling the complexities of fire-vegetation interactions and comprehending wildfire dynamics at multiple scales (Harrison *et al.,* 2021). Among various approaches, physically based models that incorporate convection and diffusion mechanisms into the dynamics of combustible forest material (CFM) provide particularly promising avenues for accurate predictions.

The present study focuses on analysing a specific class of physically based wildfire propagation models, with particular emphasis on convection and diffusion effects within the volume fraction of moisture. We address a fundamental question, whether the model admits a unique solution by employing the Lipschitz continuity approach under theorems described by Ayeni (1978). Through rigorous investigation of the existence and uniqueness of

https://dx.doi.org/10.4314/swj.v20i1.43

solutions, this research targets to contribute to the advancement of reliable predictive tools for understanding and mitigating wildfire behaviour, ultimately supporting more effective risk management strategies in fire-prone regions.

MATERIAL AND METHODS

Model formulation

Here, we consider a 1D wildfire spread model with temperature dependence of the rate of chemical reaction $K(T)$, diffusion coefficients $D_m(T)$ and $D_{ox}(T)$, and thermal conductivity k_T given respectively by

$$\left.\begin{aligned} &K(T)=k_i \exp\left(-\left(\frac{E_i}{RT}\right)\right),\ i=1,2,3,\\ &D_m = D_{m_0}\left(\frac{T}{T_0}\right),\ D_{ox}=D_{ox_0}\left(\frac{T}{T_0}\right),\\ &k_T = k_0\left(\frac{T}{T_0}\right) \end{aligned}\right\} \quad (2)$$

Where E_i the activation energy, T the temperature, R the universal gas constant, k_i the pre-exponential factor, $D_m(T)$, $D_{ox}(T)$ and k_T are moisture diffusion coefficient, oxygen diffusion coefficient and thermal conductivity respectively.

The model is formulated based on balance equations for energy and fuel, where the fuel loss due to burning corresponds to the fuel reaction rate. Convection and diffusion of moisture are considered, neglecting the ash phase with thermal equilibrium between the solid and gas phase. An existing model is discussed in Barovik and Taranchuk (2023). The leading governing equations for this investigation follow thus:

$$\left.\begin{aligned} &\frac{\partial \varphi_s}{\partial t'} + k_1\varphi_s \exp\left(-\frac{E_1}{RT}\right)=0\\ &\frac{\partial \varphi_m}{\partial t'} + v'\nabla\varphi_m + k_2\varphi_m T^{(0.5)}\exp\left(-\frac{E_2}{RT}\right)=\nabla\left(D_m\nabla\varphi_m\right)\\ &\rho_c\frac{\partial \varphi_c}{\partial t'} - \alpha_c k_1\rho_s\varphi_s\exp\left(-\frac{E_1}{RT}\right)+\frac{M_c}{M_1}k_3S_\sigma\rho_g\varphi_c C_{ox}\exp\left(-\frac{E_3}{RT}\right)=0 \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} &\rho_g\left(\frac{\partial C_{ox}}{\partial t'}+v'\nabla C_{ox}\right)+\frac{\alpha}{C_{pg}\Delta h}\left(C_{ox}-C_{ox_\infty}\right)=\nabla\left(\rho_g D_{ox}\nabla C_{ox}\right)-\left(1-\alpha_c\right)k_1\rho_s\varphi_s C_{ox}\exp\left(-\frac{E_1}{RT}\right)\\ &-k_2\rho_m T^{(0.5)}\varphi_m C_{ox}\exp\left(-\frac{E_2}{RT}\right)-k_3S_\sigma\rho_g\left(1+\frac{M_c}{M_1}C_{ox}\right)\varphi_c C_{ox}\exp\left(-\frac{E_3}{RT}\right). \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} &\left(\phi\rho_g C_{pg}+(1-\phi)\sum_{i=1}^{s+m+c}\rho_i C_{pi}\varphi_i\right)\frac{\partial T}{\partial t'}+\rho_g C_{pg}v'\nabla T+\frac{\alpha}{\Delta h}\left(T-T_\infty\right)=\nabla\left(k_T\nabla T\right)\\ &-4K_R\sigma T^4-k_2\rho_m q_2 T^{(0.5)}\varphi_m\exp\left(\frac{-E_2}{RT}\right)+k_3S_\sigma\rho_g q_3\varphi_c C_{ox}\exp\left(\frac{-E_3}{RT}\right) \end{aligned}\right\} \quad (5)$$

The initial and boundary conditions are specified as:

$$\left.\begin{aligned} &\varphi_s\big|_{t=0}=\varphi_{s0},\ \varphi_m\big|_{t=0}=\varphi_{m0},\ \varphi_c\big|_{t=0}=,\ \varphi_{c0},\ C_{ox}\big|_{t=0}=C_{ox0},\ T\big|_{t=0}=T_0;\\ &\left.\frac{\partial\varphi_m}{\partial x'}\right|_{x'=0}=0,\quad -\left(D_m^*\left.\frac{\partial\varphi_m}{\partial x'}\right|_{x'=L}+k_{mm}\,\varphi_m\big|_{x'=L}\right)=0;\\ &\left.\frac{\partial C_{ox}}{\partial x'}\right|_{x'=0}=0,\quad -\left(D_{ox}^*\left.\frac{\partial C_{ox}}{\partial x'}\right|_{x'=L}+k_{mox}\left(C_{ox}\big|_{x'=L}-C_{ox\infty}\right)\right)=0;\\ &\left.\frac{\partial \mathrm{T}}{\partial x'}\right|_{x'=0}=0,\quad -\left(k^*\left.\frac{\partial \mathrm{T}}{\partial x'}\right|_{x'=L}+h\left(T\big|_{x'=L}-T_0\right)\right)=0. \end{aligned}\right\} \quad (6)$$

Here, (3), (4) and (5) denotes the combustible foresee materials (CFMs), mass concentration of oxygen and energy (heat) equations respectively. The CFM comprises of volume fractions of dry organic matter, moisture and coke.

Where;

$\left.\begin{matrix}\varphi_s\\ \varphi_m\\ \varphi_c\end{matrix}\right\}$ are the volume fractions of dry organic substance (matter), moisture and condensed pyrolysis product

- C_{ox} is the oxygen concentration
- T is the temperature (in Kelvin)
- T_0 is the characteristics value of temperature
- x' is the dimensional coordinate in the system of coordinates connected with the center of an initial fire (distance/space)
- t' is the dimensional time
- T_∞ is the unperturbed ambient temperature
- C_{ox_0} is the characteristics value of oxygen concentration
- S_σ is the specific surface of the condensed product of pyrolysis (coke)
- v' is the dimensional equilibrium wind velocity vector
- v is the dimensionless equilibrium wind velocity vector
- L is the characteristic length
- k^* is the effective thermal conductivity
- D_{ox}^* is the effective oxygen diffusion coefficient

ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University
https://dx.doi.org/10.4314/swj.v20i1.43

D_m^* is the effective moisture diffusion coefficient
k_{mm} is the moisture convective mass transfer coefficient
k_{mox} is the oxygen convective mass transfer coefficient
h is the convective heat transfer coefficient
C_{ox_∞} is the unperturbed density of concentration of oxygen
ρ_i $i=(s,m,c)$ is the i^{th} phase densities of combustible forest materials CFMs
ρ_g is the density of gas phase.
Δh is the crown height
M_c is the molecular mass of carbon
M_1 is the mass of CFMs
C_{pg} is the specific heat capacity of a gas phase
q_2 & q_3 are the heat effects of processes of evaporation of burning
α is the coefficient of heat exchange between the atmosphere and a forest canopy
α_c is the coke number of CFMs
σ is the Stefan-BoltzMann constant.
K_R is the integral (absorption and scattering) attenuation coefficient, C_{p_i} is the specific heat.

RESULTS AND DISCUSSION

Method of solution

Ayeni (1978) investigated the issue of existence and uniqueness, of solution, revealing, among other findings, that these qualities are reasonably well understood. The subsequent system of parabolic equations serves as an illustration:

$$\left.\begin{aligned} \frac{\partial\phi}{\partial t} &= \Delta\phi + f\left(x,t,\phi,u,v\right), \qquad x\in R^n,\ t>0 \\ \frac{\partial u}{\partial t} &= \Delta u + g\left(x,t,\phi,u,v\right), \qquad x\in R^n,\ t>0 \\ \frac{\partial v}{\partial t} &= \Delta v + h\left(x,t,\phi,u,v\right), \qquad x\in R^n,\ t>0 \end{aligned}\right\}. \tag{7}$$

and,

$$\left.\begin{aligned} \phi(x,0) &= f_0(x) \\ u(x,0) &= g_0(x) \\ v(x,0) &= h_0(x) \\ x &= (x_1,x_2,...,x_n) \end{aligned}\right\}. \tag{8}$$

(S.1): $f_0(x), g_0(x), and\ h_0(x)$ are bounded for $x\in R^n$. Each has at most a countable number of discontinuities.

(S.2): f,g,h satisfies the uniform Lipschitz condition, such that,

$$\left|\varphi(x,t,\phi_1,u_1,v_1)-\varphi(x,t,\phi_2,u_2,v_2)\right| \le M\left(|\phi_1-\phi_2|+|u_1-u_2|+|v_1-v_2|\right),\ \ (x,t)\in G \tag{9}$$

Where,

$$G=\left\{(x,t): x\in R^n, 0<t<\tau\right\} \tag{10}$$

Theorem Ayeni (1978)

Let $f_0(x), g_0(x), and\ h_0(x)$ and f,g,h satisfy (S.1) and (S.2) respectively, then there exist a solution of problem (7) satisfying (8).

Dimensionless analysis

Dimensionless variables are been introduced as:

$$\left.\begin{aligned} &x=\frac{x'}{L},\ \ t=\frac{Ut'}{L},\ \ v=\frac{v'}{U},\ \ \psi_1=\frac{\varphi_s}{\varphi_{so}},\ \ \psi_2=\frac{\varphi_m}{\varphi_{mo}},\ \ \psi_3=\frac{\varphi_c}{\varphi_{co}},\ \ \phi=\frac{C_{ox}-C_{ox_\infty}}{C_{ox_0}-C_{ox_\infty}} \\ &\in=\frac{RT_0}{E},\ \ \theta=\frac{E(T-T_0)}{RT_0^2},\ \ a=\frac{E_1}{E_3},\ \ b=\frac{E_2}{E_3}. \end{aligned}\right\} \tag{11}$$

Using (11), on (3) – (6) gives the dimensionless form of the model equations (12) – (14)

$$\left.\begin{aligned} &\frac{\partial\psi_1}{\partial t} = -B_1\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right), \\ &\psi_1|_{t=0}=1 \\ &\frac{\partial\psi_2}{\partial t}+v\frac{\partial\psi_2}{\partial x} = D_1\frac{\partial}{\partial x}\left((1+\in\theta)\frac{\partial\psi_2}{\partial x}\right)-B_2\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right) \\ &\psi_2|_{t=0}=1,\ \ \left.\frac{\partial\psi_2}{\partial x}\right|_{x=0}=0,\ \ \left.\frac{\partial\psi_2}{\partial x}\right|_{x=1}=-Sh_m\psi_2(1,t) \\ &\frac{\partial\psi_3}{\partial t} = B_3\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right)-B_4\left((\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\right) \\ &\psi_3|_{t=0}=1 \end{aligned}\right\} \tag{12}$$

Science World Journal Vol. 20(No 1) 2025
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University
https://dx.doi.org/10.4314/swj.v20i1.43

$$\left.\begin{array}{l}\frac{\partial\phi}{\partial t}+v\frac{\partial\phi}{\partial x}=D_2\frac{\partial}{\partial x}\left((1+\in\theta)\frac{\partial\phi}{\partial x}\right)-B_6\phi-B_7\psi_1(\phi+B_5)\exp\left(\frac{a\theta}{(1+\in\theta)}\right)\\ -B_8(1+\in\theta)^{\frac{1}{2}}\psi_2(\phi+B_5)\exp\left(\frac{b\theta}{(1+\in\theta)}\right)-B_9(\phi+B_{10})(\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\\ \phi|_{t=0}=1,\quad \left.\frac{\partial\phi}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\phi}{\partial x}\right|_{x=1}=-Sh_{ox}\phi(1,t)\end{array}\right\}$$

(13)

$$\left.\begin{array}{l}\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\left(\lambda_T(1+\in\theta)\frac{\partial\theta}{\partial x}\right)-B_{11}(\theta+B_{12})-R_a(1+4\in\theta)\\ -\delta_1\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right)+\delta_2\psi_3(\phi+B_5)\exp\left(\frac{\theta}{(1+\in\theta)}\right)\\ \theta|_{t=0}=0,\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=1}=-Nu\theta(1,t)\end{array}\right\}$$

(14)

where,

R_a is Radiation number

P_{emj}, $j=1,2$ are Peclet mass numbers

P_e is the peclet energy number

δ_i , $i=1,2$ are Frank-Kamenetskii numbers

Sh_m is the Sherwood number (moisture)

Sh_{ox} is the Sherwood number (oxidizer)

Nu is Nusselt number.

Existence and Uniqueness of Solution

Here, (12) – (14) are written as follows, with expansion effect on volume fraction of moisture, mass concentration of oxygen and energy equations

$$\left.\begin{array}{l}\frac{\partial\psi_1}{\partial t}=-B_1\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right),\\ \psi_1|_{t=0}=1\\ \frac{\partial\psi_2}{\partial t}+v\frac{\partial\psi_2}{\partial x}=D_1\frac{\partial^2\psi_2}{\partial x^2}+\in D_1\theta\frac{\partial^2\psi_2}{\partial x^2}+\in D_1\frac{\partial\theta}{\partial x}\frac{\partial\psi_2}{\partial x}-B_2\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right)\\ \psi_2|_{t=0}=1,\quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=1}=-Sh_m\psi_2(1,t)\\ \frac{\partial\psi_3}{\partial t}=B_3\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right)-B_4\left((\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\right)\\ \psi_3|_{t=0}=1\end{array}\right\}$$

(15)

$$\left.\begin{array}{l}\frac{\partial\phi}{\partial t}+v\frac{\partial\phi}{\partial x}=D_2\frac{\partial^2\phi}{\partial x^2}+\in D_2\theta\frac{\partial^2\phi}{\partial x^2}+\in D_2\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}-B_6\phi-B_7\psi_1(\phi+B_5)\exp\left(\frac{a\theta}{(1+\in\theta)}\right)\\ -B_8(1+\in\theta)^{\frac{1}{2}}\psi_2(\phi+B_5)\exp\left(\frac{b\theta}{(1+\in\theta)}\right)-B_9(\phi+B_{10})(\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\\ \phi|_{t=0}=1,\quad \left.\frac{\partial\phi}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\phi}{\partial x}\right|_{x=1}=-Sh_{ox}\phi(1,t)\end{array}\right\}$$

(16)

$$\left.\begin{array}{l}\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial x}=\lambda_T\frac{\partial^2\theta}{\partial x^2}+\in\lambda_T\theta\frac{\partial^2\theta}{\partial x^2}+\in\lambda_T\left(\frac{\partial\theta}{\partial x}\right)^2-B_{11}(\theta+B_{12})-R_a(1+4\in\theta)\\ -\delta_1\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right)+\delta_2\psi_3(\phi+B_5)\exp\left(\frac{\theta}{(1+\in\theta)}\right)\\ \theta|_{t=0}=0,\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=1}=-Nu\theta(1,t)\end{array}\right\}$$

(17)

Theorem

Suppose $|\psi_1|\le h_1, |\psi_2|\le h_2, |\psi_3|\le h_3, |\phi|\le h_4,$

$\left|\frac{\partial^2\psi_2}{\partial x^2}\right|\le h_5, \left|\frac{\partial^2\phi}{\partial x^2}\right|\le h_6, \left|\frac{\partial^2\theta}{\partial x^2}\right|\le h_7$

Then equation (15) – (17) have unique solution.
In the proof we shall employ the Theorem 3.1

Proof of Theorem

Rewriting the equations (15) – (17) as system of equations thus;

$$\left.\begin{array}{l}\frac{\partial\psi_1}{\partial t}=g_1(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta),\ x\in R^n,\ t>0\\ \frac{\partial\psi_2}{\partial t}+v\frac{\partial\psi_2}{\partial x}=D_1\frac{\partial^2\psi_2}{\partial x^2}+g_2(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta),\ x\in R^n,\ t>0\\ \frac{\partial\psi_3}{\partial t}=g_3(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta),\ x\in R^n,\ t>0\end{array}\right\}$$

(18)

$$\frac{\partial\phi}{\partial t}+v\frac{\partial\phi}{\partial x}=D_2\frac{\partial^2\phi}{\partial x^2}+g_4(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta),\ x\in R^n,\ t>0$$

(19)

$$\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial x}=\lambda_T\frac{\partial^2\theta}{\partial x^2}+g_5(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta),\ x\in R^n,\ t>0$$

(20)

Where

Science World Journal Vol. 20(No 1) 2025 https://dx.doi.org/10.4314/swj.v20i1.43
www.scienceworldjournal.org
ISSN: 1597-6343 (Online), ISSN: 2756-391X (Print)
Published by Faculty of Science, Kaduna State University

$$\left.\begin{aligned} g_1(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta) &= -B_1\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right)\\ g_2(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta) &= \in D_1\theta\frac{\partial^2\psi_2}{\partial x^2}+\in D_1\frac{\partial\theta}{\partial x}\frac{\partial\psi_2}{\partial x}-B_2\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right)\\ g_3(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta) &= B_3\psi_1\exp\left(\frac{a\theta}{(1+\in\theta)}\right)-B_4(\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\end{aligned}\right\}$$

(21)

$$\left.\begin{aligned} g_4(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta) &= \in D_2\theta\frac{\partial^2\phi}{\partial x^2}+\in D_2\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial x}-B_6\phi-B_7\psi_1(\phi+B_5)\exp\left(\frac{a\theta}{(1+\in\theta)}\right)\\ &-B_8(1+\in\theta)^{\frac{1}{2}}\psi_2(\phi+B_5)\exp\left(\frac{b\theta}{(1+\in\theta)}\right)-B_9(\phi+B_{10})(\phi+B_5)\psi_3\exp\left(\frac{\theta}{(1+\in\theta)}\right)\end{aligned}\right\}$$

(22)

$$\left.\begin{aligned} g_5(x,t,\psi_1,\psi_2,\psi_3,\phi,\theta) &= \in\lambda_T\theta\frac{\partial^2\theta}{\partial x^2}+\in\lambda_T\left(\frac{\partial\theta}{\partial x}\right)^2-B_{11}(\theta+B_{12})-R_a(1+4\in\theta)\\ &-\delta_1\psi_2(1+\in\theta)^{\frac{1}{2}}\exp\left(\frac{b\theta}{(1+\in\theta)}\right)+\delta_2\psi_3(\phi+B_5)\exp\left(\frac{\theta}{(1+\in\theta)}\right)\end{aligned}\right\}$$

(23)

According to Toki and Tokis (2007), the fundamental solutions of equation (18) – (20) are as follows:

$$\left.\begin{aligned} G_1(x,t) &= C_1\\ G_2(x,t) &= \frac{x}{2\pi^{\frac{1}{2}}(D_1)^{\frac{1}{2}}t^{\frac{3}{2}}}\exp\left(\frac{v}{2D_1}x-\frac{v^2}{4D_1}t-\frac{x}{4D_1t}\right)\\ G_3(x,t) &= C_2\end{aligned}\right\}$$

(24)

$$G_4(x,t)=\frac{x}{2\pi^{\frac{1}{2}}(D_2)^{\frac{1}{2}}t^{\frac{3}{2}}}\exp\left(\frac{v}{2D_2}x-\frac{v^2}{4D_2}t-\frac{x}{4D_2t}\right)$$

(25)

$$G_5(x,t)=\frac{x}{2\pi^{\frac{1}{2}}(\lambda_T)^{\frac{1}{2}}t^{\frac{3}{2}}}\exp\left(\frac{v}{2\lambda_T}x-\frac{v^2}{4\lambda_T}t-\frac{x}{4\lambda_Tt}\right)$$

(26)

Next, it suffices to show that the Lipschitz condition in Theorem 3.1 is satisfied. That is, if we are able to show that;

$$\left.\begin{aligned} &\left|g_i(x,t,\psi_{11},\psi_{21},\psi_{31},\phi_1,\theta_1)-g_i(x,t,\psi_{12},\psi_{22},\psi_{32},\phi_2,\theta_2)\right|\le\\ &k_i\left(|\psi_{11}-\psi_{12}|+|\psi_{21}-\psi_{22}|+|\psi_{31}-\psi_{32}|+|\phi_1-\phi_2|+|\theta_1-\theta_2|\right),\\ &i=1,2,\ldots,5.\end{aligned}\right\}$$

(27)

It is important to note that:

$$k_i=\max\left\{\left|\frac{\partial g_i}{\partial\psi_1}\right|,\left|\frac{\partial g_i}{\partial\psi_2}\right|,\left|\frac{\partial g_i}{\partial\psi_3}\right|,\left|\frac{\partial g_i}{\partial\phi}\right|,\left|\frac{\partial g_i}{\partial\theta}\right|\right\},\quad i=1,2,\ldots,5$$

(28)

Then,

$$\left|\frac{\partial g_1}{\partial\psi_1}\right|=\left|-\left(B_1e^{\frac{a\theta}{(1+\in\theta)}}\right)\right|\le B_1e^{\frac{a}{\in}},\quad 0\le\theta<\infty,$$

$$\left|\frac{\partial g_1}{\partial\psi_2}\right|=0,\quad\left|\frac{\partial g_1}{\partial\psi_3}\right|=0,\quad\left|\frac{\partial g_1}{\partial\phi}\right|=0,$$

$$\left|\frac{\partial g_1}{\partial\theta}\right|=\left|-\left(\frac{1}{1+\in\theta}\right)^2\left(aB_1e^{\frac{a\theta}{(1+\in\theta)}}\right)\psi_1\right|\le(aB_1)h_1,$$

$$\left|\frac{\partial g_2}{\partial\psi_1}\right|=0,$$

$$\left|\frac{\partial g_2}{\partial\psi_2}\right|=\left|-B_2(1+\in\theta)^{\frac{1}{2}}e^{\frac{b\theta}{(1+\in\theta)}}\right|\le B_2e^{\frac{b}{\in}},$$

$$\left|\frac{\partial g_2}{\partial\psi_3}\right|=0,\quad\left|\frac{\partial g_2}{\partial\phi}\right|=0,$$

$$\left|\frac{\partial g_2}{\partial\theta}\right|=\left|\in D_1\frac{\partial^2\psi_2}{\partial x^2}-B_2\psi_2(1+\in\theta)^{-\frac{1}{2}}e^{\frac{b\theta}{(1+\in\theta)}}\left(b+\frac{\in}{2}\right)\right|\le\left(\in D_1h_5+B_2h_2\left(b+\frac{\in}{2}\right)\right),$$

$$\left|\frac{\partial g_3}{\partial\psi_1}\right|=\left|-\left(B_3e^{\frac{a\theta}{(1+\in\theta)}}\right)\right|\le B_3e^{\frac{a}{\in}},\quad\left|\frac{\partial g_3}{\partial\psi_2}\right|=0,$$

$$\left|\frac{\partial g_3}{\partial\psi_3}\right|=\left|-B_4(\phi+B_5)e^{\frac{\theta}{(1+\in\theta)}}\right|\le B_4(h_4+B_5)e^{\frac{1}{\in}},$$

$$\left|\frac{\partial g_3}{\partial\phi}\right|=\left|-B_4\psi_3e^{\frac{\theta}{(1+\in\theta)}}\right|\le B_4h_3e^{\frac{1}{\in}},$$

$$\left|\frac{\partial g_3}{\partial\theta}\right|=\left|\left(\frac{1}{1+\in\theta}\right)^2\left(aB_3\psi_1e^{\frac{a\theta}{(1+\in\theta)}}-B_4(\phi+B_5)\psi_3e^{\frac{a\theta}{(1+\in\theta)}}\right)\right|\le\left(aB_3h_1+B_4(h+B_5)h_3\right),$$

$$\left|\frac{\partial g_4}{\partial\psi_1}\right|=\left|-B_7(\phi+B_5)e^{\frac{a\theta}{(1+\in\theta)}}\right|\le B_7(h_4+B_5)e^{\frac{a}{\in}},$$

$$\left|\frac{\partial g_4}{\partial\psi_2}\right|=\left|-B_8(1+\in\theta)^{\frac{1}{2}}(\phi+B_5)e^{\frac{b\theta}{(1+\in\theta)}}\right|\le B_8(h_4+B_5)e^{\frac{b}{\in}},$$

$$\left|\frac{\partial g_4}{\partial\psi_3}\right|=\left|-B_9(\phi+B_{10})(\phi+B_5)e^{\frac{\theta}{(1+\in\theta)}}\right|\le B_9(h_4+B_5)(h_4+B_{10})e^{\frac{1}{\in}},$$

$$\left|\frac{\partial g_4}{\partial\phi}\right|=\left|-\left(B_6+B_7\psi_1e^{\frac{a\theta}{(1+\in\theta)}}+B_8(1+\in\theta)^{\frac{1}{2}}\psi_2e^{\frac{b\theta}{(1+\in\theta)}}-B_9(2\phi+B_{10}+B_5)\psi_3e^{\frac{\theta}{(1+\in\theta)}}\right)\right|$$

$$\le\left(B_6+B_7h_1e^{\frac{a}{\in}}+B_8h_2e^{\frac{b}{\in}}+2B_9h_4+B_9(B_{10}+B_5)\right),$$

$$\left|\frac{\partial g_4}{\partial\theta}\right|=\left|-\left(B_8(1+\in\theta)^{-\frac{1}{2}}(\phi+B_5)\psi_2e^{\frac{b\theta}{(1+\in\theta)}}\left(\frac{\in}{2}+b\right)+B_9\left(\frac{1}{1+\in\theta}\right)^2(\phi+B_{10})(\phi+B_5)\psi_3e^{\frac{\theta}{(1+\in\theta)}}\right)\right|$$

$$\le\left(B_8(h_4+B_5)h_2\left(\frac{\in}{2}+b\right)+B_9(h_4+B_5)(h_4+B_{10})h_3\right),$$

$$\left|\frac{\partial g_5}{\partial \psi_1}\right| = 0, \quad \left|\frac{\partial g_5}{\partial \psi_2}\right| = 0,$$

$$\left|\frac{\partial g_5}{\partial \psi_3}\right| = \left|-\delta_2\left(\phi + B_5\right)e^{\frac{\theta}{(1+\in\theta)}}\right| \le \delta_2\left(h_2 + B_5\right)e^{\frac{1}{\in}},$$

$$\left|\frac{\partial g_5}{\partial \phi}\right| = \left|\delta_2 \psi_3 e^{\frac{\theta}{(1+\in\theta)}}\right| \le \delta_2 h_3 e^{\frac{1}{\in}},$$

$$\left|\frac{\partial g_5}{\partial \theta}\right| = \left| \begin{matrix} \in \lambda_T \dfrac{\partial^2 \theta}{\partial x^2} - B_{11} - 4R_a \in - \begin{pmatrix} \dfrac{\delta_1 \psi_2}{2} \in \left(1+\in\theta\right)^{-\frac{1}{2}} e^{\frac{b\theta}{(1+\in\theta)}} \\ +\delta_1 \psi_2 \left(1+\in\theta\right)^{-\frac{1}{2}} b e^{\frac{b\theta}{(1+\in\theta)}} \end{pmatrix} + \\ \dfrac{\delta_2 \psi_3 \left(\phi + B_5\right)}{\left(1+\in\theta\right)} e^{\frac{\theta}{(1+\in\theta)}} \end{matrix} \right| \le \begin{pmatrix} \in \lambda_T h_7 + B_{11} + 4R_a \in + \delta_1 h_2 \left(\dfrac{\in}{2} + b\right) \\ +\delta_2 h_3 \left(h_4 + B_5\right) \end{pmatrix}$$

Hence, $k_1 = B_1 e^{\frac{a}{\in}}, \; k_2 = B_2 e^{\frac{b}{\in}},$

$$k_3 = B_4\left(h_4 + B_5\right)e^{\frac{1}{\in}},$$

$$k_4 = \left(B_6 + B_7 h_1 e^{\frac{a}{\in}} + B_8 h_2 e^{\frac{b}{\in}} + 2B_9 h_4 + B_9\left(B_{10} + B_5\right)\right),$$

$$k_5 = \begin{pmatrix} \in \lambda_T h_7 + B_{11} + 4R_a \in + \delta_1 h_2 \left(\dfrac{\in}{2} + b\right) \\ +\delta_2 h_3 \left(h_4 + B_5\right) \end{pmatrix}$$

Clearly, $g_i\left(x, t, \psi_1, \psi_2, \psi_3, \phi, \theta\right), \;\; i = 1, 2, ..., 5$ are Lipschitz continuous. Hence by Theorem 3.1, the result follows. This completes the proof.

Conclusions
In this study, we analytically establish the existence and uniqueness of solutions to the governing model equation for wildfire spread. A key novelty of our work is the incorporation of convection and diffusion terms into the volume fraction of moisture, which, to the best of our knowledge, has not been previously integrated in this manner. By explicitly accounting for the transport and spatial distribution of moisture within combustible forest materials, we provide a more comprehensive framework for modeling wildfire dynamics. Our findings offer a solid theoretical foundation for future numerical simulations and reinforce the well-posedness of the model, ensuring its ability to accurately capture the underlying physical phenomena under specified conditions and assumptions.

Author Contributions: ABZ conceptualized the article and conducted the literature search. ABZ and OAA developed and modified the model, while all authors critically revised the work. ABZ, PA, and OAA jointly conceived and designed the study framework. ABZ and OAA performed the problem-solving, with ABZ drafting the initial manuscript. OAA supervised the manuscript throughout the process. PA and OAA carried out the final review and revision of the paper.

Funding No: funding was received to assist with the preparation of this manuscript.

Declarations:

Conflict of interest: The authors have no conflict of interest to declare that are relevant to the content of this article.

Data availability: Not applicable

REFERENCES
Ayeni, R. O. (1978). *Thermal Runaway*. Unpublished PhD Thesis, Cornell University, USA.
Barovik, D. and Taranchuk, V. (2023). Surface forest fires modelling: temperature and oxygen dynamics near fuelbreaks. Baltic Journal of Modern Computing, vol. 11(2), pp. 226-240, https://doi.org/10.22364/bjmc.2023.11.2.01
Crompton, O. V., Boisrame, G. F. S., Rakhmatulina, E., Stephens, S. L. and Thompson, S. E. (2022). Fire return intervals explain different vegetation cover responses to wildfire restoration in two Sierra Nevada basins. For. Ecol. Manag. 521 (2022) 120429
Feckan, M and Pacuta, J (2018). Existence of solution of a forest fire spread model. Applied Mathematics Letters, vol. 83, pp. 227-231.
Harrison, S. P., Prentice, I. C., Bloomfield, K. J., Dong, N., Forkel, M., Forrest, M., Ningthoujam, R. K., Pellegrini, A., Shen, Y., Baudena, M., Cardoso, A. W., Huss, J. C., Joshi, J., Oliveras, I., Pausas, J. G. and Simpson, K. J. (2021). Understanding and modelling wildfire regimes: an ecological perspective, Environ. Res. Lett. 16 (2021) 125008.
Kahanji, C., Walls, R. S. and Cicione, A. (2019). Fire spread analysis for the 2017 Imizamo Yethu informal settlement conflagration in South Africa. International Journal of Disaster Risk Reduction, vol. 39, pp. 1-12, https://doi.org/10.1016/j.ijdrr.2019.101146
Mangiameli, M., Mussumeci, G. and Cappello, A. (2021). Forest fire spreading using free and open-source gis technologies. Geomatics vol. 1, pp. 50–64. https://doi.org/10.3390/geomatics1010005.
Morgan, A. G. (2024). Existence-Uniqueness Theory and Small-Data Decay for a Reaction-Diffusion Model of Wildfire Spread. [math.AP] https://arxiv.org/html/2406.00575v1
Mitra, K., Peng, Q. and Reisch, C. (2024). Studying wildfire fronts using advection-diffusion-reaction models. Eindhoven University of Technology, [math.AP], https://research.tue.nl/en/publications/studying-wildfire-fronts-using-advection-diffusion-reaction-model.
Senande-Rivera, M., Insua-Costa, D. and Miguez-Macho, G. (2022). Spatial and temporal expansion of global wildland fire activity in response to climate change. Nat Commun 13, 1208 (2022). doi:10.1038/s41467-022-28835-2.
Toki, C. J. and Tokis, J. N. (2007). Exact solutions for the unsteady free convection flows on a porous plate with time-dependent heating. ZAMM · Z. Angew. Math. Mech. 87(1), pp. 4 – 13, **DOI** 10.1002/zamm.200510291.