SHORT COMMUNICATION REPORT

MULTISETS AND INEQUALITIES

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A Multiset (mset) is an unordered collection of objects (called elements) in which unlike a standard (cantorian) set, elements are allowed to repeat. In other words an mset is a structure in which elements may appear more than once and hence non cantorian.

Among the failures of some cantorian set theorems on msets, Cantor's Theorem on the Cardinality of a powerset failed on msets. That is 'given an mset u such that u is not a Cantorian set', then the inequality;

$$|\wp(u)| < 2^{|U|} \qquad (1)$$

holds, where $|\ |$ denotes the Cardinality and $\wp(u)$ the powerset of u. However (Singh & Singh 2003), in their effort to determine the cardinality of a powerset (following the failure of Cantor's theorem) established the cardinality of the powerset of an mset. In this paper we establish the proof of certain inequalities on the set of natural numbers based on the Cardinality of a powerset of an mset starting with the concept background of msets in section two. In section three we review some basic definitions and set up a pace for the proof of such inequalities in section four and in conclusion we establish some inequalities on the set of natural numbers in section five.

The concept background of multisets: In contrast to classical (Cantorian) set theory in which an element cannot appear more than once, an mset is an unordered collection of objects into a whole in which certain or all elements are allowed to repeat (finitely for almost all known application areas). However, in a purely theoretical development, infinite multiplicities of elements in an mset are dealt with (Singh & Singh 2003). The number of 'copies' of an element, treated without any preference, appearing in an mset is called its Multiplicity. The Cardinality of an mset is the sum of the Multiplicities of its elements. A considerable amount of effort has also gone into decoding the dark realm of negative Multiplicities (Blizard 1990). Notationally, an mset containing one occurrence of a, two occurrence of b and three occurrence of c is formally written as

$[[a,b,b,c,c,c]]$ or $[a,b,b,c,c,c]$ or $[a,b,c]_{1,2,3}$ or $[a^1,b^2,c^3]$ or $[1.a,2.b,3.c]$ or $[a.1,b.2,3.c]$ depending on one's taste and convenience in a particular context (for example, if a,b,c are themselves numbers).

Also, $[a,b,c]_{1,2,3} = [c,a,b]_{3,1,2}$, and $[a]_1 = \{a\}$.

As these results concern the number of Multisubset (msubset, for short), of an mset as described in (Blizard 1989), we outline an explanation of msubset relation '$\subseteq$'; henceforth called BMST for convenience. BMST is a two-sorted first order theory augmented as a generalization of classical ZFC (Zormelo-Fraenkel plus axiom of choice) set theory and proved consistent relative to ZFC. A model is constructed by interpreting $x \in^n y$ as $y(x) = n$.That is, msets are modelled by integer-valued functions. Besides the logical symbols $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\exists$ (there exist) $\forall$ (for all). BMST adopts the following two types of variables: Mset variables denoted by $x, y, z, \ldots$ with M, the collection for msets as universe and numeric variables, denoted by $k,l,m,n,\ldots$ with N, the set of natural numbers as the universe.

Two sorts of equality relation $=_M$, for mset variables and of sorting $M \times M$ $=_N$ for numeric variables and of sorting $N \times N$. The metavariables u,v and w are used to denote mset terms and s,t to denote numeric terms. A primitive ternary membership relation 'e' of sorting $M \times M \times N$ is used to denote 'membership with multiplicity'.

Thus, if u,v are msets terms and t a numeric term, then the atomic formula $e(u,v,t)$ is read 'u belongs to v with multiplicity t'. In order to be close to the intended meaning, $e(u,v,t)$ is replaced by $u \in^t v$. Also '$u \in v$' standing for $(\exists n)(u \in^n v)$ and correspondingly '$u \notin v$' standing for $\neg(\exists n)(u \in^n v)$ or $(\forall n)\neg(u \in^n v)$ are introduced as well formed formulae.

The entire theory of Peano arithmetic is assumed for characterizing the numeric universe of BMST. Among the mset axioms of BMST, we shall be concerned with the following in this paper;

The axiom of exact multiplicity:
$\forall x \forall y \forall n \forall m((x \in^n y \wedge x \in^m y) \rightarrow n = m)$. That is, the multiplicity with which an element belongs to an mset is unique.

The axiom of Extensionality:
$\forall x \forall y(\forall z \forall n(z \in^n x \leftrightarrow z \in^n y) \rightarrow x = y)$ (Blizard 1989).

The axiom of empty mset: $\exists y \forall x \forall n \neg(x \in^n y)$.This axiom establishes the existence of a unique empty mset normally denoted by $\varnothing$

The powerset axiom
$\forall z \forall x \exists y (Set(y) \wedge \forall z (z \in y \leftrightarrow z \subseteq x))$. In other words, for every mset x, there is a set whose elements are exactly the msubsets of x. The set y is unique by axiom (V) of BMST (Blizard 1989). For any mset term u, we denote the set of all msubsets of u by $\wp(u)$, and we call it the powerset of u. For example,
$\wp([x,y]_{3,1}) = \{\varnothing, \{x\}, [x]_2, [x]_3, \{y\}, \{x,y\}, [x,y]_{2,1}, [x,y]_{3,1}\}$.

SOME BASIC DEFINITIONS.
$Set(u)$: The term $Set(u)$ stands for
$u = \varnothing \vee \forall x \forall n (x \in^n u \rightarrow n = 1)$.

Hence, as $Set(\varnothing)$, the mset $\varnothing$ is called the empty set. Using the axiom of specification (Blizard 1989), it can be shown that for every mset x, there corresponds a unique mset x^* (called its root (support) set) containing all the distinct elements of x. Notes that one or more elements in a root set of an mset may be msets themselves.

msubset relation '$\subseteq$'.: For all mset terms u, v, $u \subseteq v$ (u is an msubset of v) if $\forall z \forall n (z \in^n u \rightarrow \exists m (n \le m \wedge z \in^m v))$. Or $\forall z \forall n (z \in^n u \rightarrow z \in^n_+ v)$ where $z \in^n_+ v$ implies 'z belongs to v at least n times', however $z \in^n u$ means z belongs to u exactly n times' (Singh 2006).

In relation to its msubsets, an mset is called the parent mset. For example, v in the above definition is the parent mset of u. The relation '$\subseteq$' is reflexive, antisymmetric, and transitive (i.e, it is a partial ordering).

It is important to observe that defining '$\subseteq$' as '$x \subseteq y$ if $\forall z (z \in x \rightarrow z \in y)$' (classical definition) will be counter intuitive. It is not antisymmetric and does not preserve the strong intuitive requirements in this regard that the Cardinality of an msubset be atmost the Cardinality of its parent mset. A similar argument rules out an equivalent definition of '$\subseteq$' as '$x \subseteq y$ iff $x^* \subseteq y^*$'. However, in this regard, the notion of 'similar' msets is introduced as follows:

Msets u, v are said to be similar if $\forall z (z \in u \leftrightarrow z \in v)$.
Clearly similar msets need not be equal msets.

An mset u is:

(i) a part of v if $u \subseteq v \wedge u \ne \varnothing \wedge u \ne v$
(ii) whole msubset of v if $\forall z \forall n (z \in^n u \rightarrow z \in^n v)$,
(iii) a full msubset of v if
$u \subseteq v \wedge \forall z (z \in v \rightarrow z \in u)$

ESTABLISHED RESULTS.
For any mset $u = [x_1, x_2, ..., x_n]_{k_1, k_2, ..., k_n}$ such that
$k_i \in \mathrm{N} \wedge 1 < k_i < +\infty, i = 1, 2, 3, ...$

(i) $|\wp(u)| < 2^s$ where $s = \sum_{i=1}^{n} k_i$ (Blizard 1989).

(ii) Let $\wp w(u)$ be the set of all whole msubsets of u, then

$|\wp w(u)| = 2^n$ (Singh & Singh 2003).

(iii).Let $\wp f(u)$ be the set of all full msubsets of u, then

$|\wp f(u)| = \prod_{i=1}^{n} k_i$.

(iv) $|\wp(u)| = \prod_{i=1}^{n} (1 + k_i)$ (Singh & Singh 2003).

INEQUALITIES.
With the above established results, the following inequalities was deduced:

$\forall k_i, n \in \square \wedge 1 \le k_i < +\infty, k_i \le n \; i = 1, 2, 3, 4...$

(i) $\prod_{i=1}^{n} k_i < 2^{\sum_{i=1}^{n} k_i}$ (ii) $2^n \le 2^{\sum_{i=1}^{n} k_i}$ (iii) $n \le \sum_{i=1}^{n} k_i$ (iv) $2^n \le \prod_{i=1}^{n} (1 + k_i)$

(v) $\prod_{i=1}^{n} k_i < \prod_{i=1}^{n} (1 + k_i)$ (vi) $\prod_{i=1}^{n} (1 + k_i) \le 2^{\sum_{i=1}^{n} k_i}$

Proof:
Let $v = [x_1, x_2, x_3, ..., x_n]_{k_1, k_2, k_3, ..., k_n}$
Be an mset such that, $1 \le k_i < +\infty$. let $\wp(v)$ be the powerset of v let $\wp f(v)$ be the set of all full msubsets of v and $\wp w(v)$ the set all whole msubsets of v. Clearly;

$$\wp f(v) \subset \wp(v) \qquad (i).$$

Thus;
$$|\wp f(v)| < |\wp(v)| \qquad (ii).$$

$$\wp w(v) \subseteq \wp(v) \qquad (iii).$$

Thus;

$$|\wp w(v)| \le |\wp(v)| \qquad (iv).$$

Using $(i),(ii)$ above and 4(i), we have;

$$|\wp w(v)| \leq |\wp(v)| \leq 2^{\sum_{i=1}^{n} k_i} \qquad (v)$$

and

$$|\wp f(v)| < |\wp(v)| \leq 2^{\sum_{i=1}^{n} k_i} \qquad (vi).$$

(i). Clearly from 4 (iii) and (vi) the result follows.
(ii). From 4(ii) and (v) above, the result follows.
(iii). A deduction from (ii).
(iv). This is clear from 4(ii, iv) and (iv) above
(v). The result follows from 4(iii, iv) and (vi) above.
(vi). Following 4(iv) and (vi) above.

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