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ON CONTRASTS FOR DISCRIMINATING THE OPTIMAL SEMI-LATIN SQUARES WITH SIDE SIX AND BLOCK SIZE TWO

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ABSTRACT
In this work, we investigate the contrasts properties of the optimal semi-Latin squares with side six and block size two with a view to discriminating amongst them. The Variances, as well as the average Variance of the estimate of elementary contrasts of treatments was computed for each square with the aid of MATLAB and the results compared with each other. From the results, the average variance of elementary treatment contrasts for the three squares Λ_1, Λ_2 and Λ_3 is approximately the same. Again, the squares Λ_2 and Λ_3 have the same and minimal number of distinct variances of elementary contrasts compared to Λ_1, hence preferable to Λ_1. Furthermore, Λ_3 has a minimal value of the maximum variances of contrasts amongst these squares. Thus, Λ_3 becomes the most preferable square amongst all of them; while Λ_2 is preferable to Λ_1.

Keywords: Elementary contrast, optimal design, efficiency factor, regular-graph design, quotient block design, connected design.

INTRODUCTION
An $(n \times n)/k$ semi-Latin square is a row-column design with n rows and n columns which contains nk distinct symbols $(n, k > 1)$ called treatments that appear in such a way that there are exactly k symbols in each row-column intersection called block (or cell), and each symbol occurs only once in each row and also once in each column: see, for instance, Bailey and Royle (1997), Bedford and Whitaker (2001), as well as Soicher (2013). A semi-Latin square generalizes Latin squares and has applications in areas such as Agricultural experiments, consumer testing, and human-Machine interaction: see, for, instance, Preece and Freeman (1983), Bailey (1988, 1992), Soicher (2013). A semi-Latin square with side six and block size two is indeed a $(6 \times 6)/2$ semi-Latin square.

In the choice of designs for experimentation, the optimal ones are usually sought for. Optimal designs are known to have high efficiency factors. The four widely used measures of the efficiency factor of a design include: $A-, D-, E-,$ and $E'-$measures. A design within a given class is considered optimal in that class with respect to a given criterion if it maximizes the value of the efficiency factor corresponding to such criterion among all the designs in the same class with it: see Bailey and Royle (1997).

Contrasts of treatments are important in determining the efficiency of a design. According to Bailey and Royle (1997), a design having a high efficiency factor would tend to have low variances of within-block estimators. The average of the variances of the estimates, $\hat{\tau}_i$-$\hat{\tau}_j$, of the contrasts τ_i-τ_j, for all distinct pairs (i, j) of treatments of a design gives a measure of efficiency of the design: see John (1971). Furthermore, a design is optimal if it has a minimal number of distinct pairwise treatment variances amongst all the designs in the same class with it: see Cameron *et al.* (2003).

Chigbu (1999) found three optimal $(4 \times 4)/4$ semi-Latin squares using the $A-$, $D-$, and $E-$optimality criteria. Chigbu (2003) observed that these squares differ in their treatment concurrences, which suggests inherent differences amongst them. He found the best square amongst them using an analytic approach by computing for each square and comparing the variances of elementary treatment contrasts amongst the squares; but did not further classify and/or discuss the sameness or otherwise of the other two good squares. Subsequently, Chigbu (2004) established the same square due to Chigbu (2003) as the best, while rating the other two squares as the same using a numerical approach which involves computation of a generalized inverse of the information matrix of each square. Uto and Chigbu (2010) ascertained the same square due to Chigbu (2003, 2004) as the best and further distinguished between the remaining two squares by computing and comparing the variances of the differences in concurrences among the squares.

Bailey and Royle (1997) found optimal $(6 \times 6)/2$semi-Latin squares using the $A-, D-, E-,$ and$E'-$optimality criteria which were applied to its quotient block design. For each of these criteria, they found the optimal semi-Latin square among regular-graph semi-Latin square designs of that size. Though their $E-$and $E'-$optimal designs were distinct, they found the same square to be optimal under the $A-$ and $D-$criteria

In this paper, we investigate the contrasts properties of these squares with a view to discriminating amongst them.

PRELIMINARIES
Generally, in optimal block design theory, a regular-graph design, when it exists, provides the basis for selecting optimal designs, since an optimal regular-graph design is known to be universally optimal over all the designs in the same class with it: see, for instance, Bailey and Royle (1997). The semi-Latin squares considered in this paper, were adapted from Bailey and Royle (1997) and are designated Λ_1, Λ_2 and Λ_3. They are presented in Tables 1, 2 and 3, respectively.

Table 1: The semi-Latin Square Λ_1

0 1	8 A	4 5	3 6	2 B	7 9
3 5	0 2	7 A	1 8	6 9	4 B
6 8	5 9	0 3	7 B	1 4	2 A
9 B	1 7	2 6	0 4	3 A	5 8
2 7	4 6	8 B	9 A	0 5	1 3

4 A	3 B	1 9	2 5	7 8	0 6

Table 2: The semi-Latin square Λ_2

0 5	4 A	3 8	2 7	6 9	1 B
4 9	2 8	0 7	6 B	1 A	3 5
1 8	0 6	4 B	3 A	2 5	7 9
2 B	1 7	5 A	0 9	3 4	6 8
3 6	5 B	2 9	1 4	7 8	0 A
7 A	3 9	1 6	5 8	0 B	2 4

Table 3: The semi-Latin square Λ_3

0 1	5 A	2 7	4 9	6 B	3 8
4 B	2 3	1 9	8 A	5 7	0 6
3 6	0 8	4 5	1 B	2 A	7 9
5 8	9 B	0 A	6 7	1 3	2 4
7 A	4 6	3 B	0 2	8 9	1 5
2 9	1 7	6 8	3 5	0 4	A B

Λ_1 is $A-$ and $D-$optimal among semi-Latin squares of that size, while Λ_2 and Λ_3 are $E-$ and $E^{'}-$optimal, respectively. Each of these squares is a regular-graph design, with its quotient block design having treatment concurrences equal to zero or one.

A regular-graph design has its treatment concurrences not differing in value by more than one in absolute terms: see, for instance, Cheng (1978), Paterson (1986), as well as Bailey (1992). Also, a semi-Latin square with $k = n - 1$ whose quotient block design is a regular-graph design is a Trojan square: see Bailey (1992).

We note that semi-Latin squares are usually analyzed as incomplete block designs; and in the analysis of incomplete block designs, every treatment contrast is of interest, and needs to be estimated and compared through the variance of its estimator: see Chigbu (2004). In some designs, such as the ones in this work, all the elementary contrasts are estimable from intra-block comparisons. Those designs with such attribute are thus connected designs: see, for example, Raghavarao (1971).

Definition 1 (Contrast)
A contrast in treatment parametersis a linear function (or combination) $\underline{C}'\underline{\tau}$ of these parameters whose sum of entries of the coefficient vector $\underline{C}$ is zero; where $\underline{\tau}$ is the vector of treatment parameters. It is said to be an elementary contrast if $\underline{C}$ has only two non-zero elements 1 and -1.

Elementary contrasts of treatment effects show the comparison of treatments involved in them and it is very desirable to estimate all the elementary contrasts. Fundamentally, contrasts of the form τ_i-$\tau_j (i \neq j)$ are called elementary contrasts. The best linear unbiased estimator of τ_i-τ_j is $\hat{\tau}_i$-$\hat{\tau}_j$,$\forall i = 1,2, \dots, v$: see Raghavarao (1971).

For a block design with nk treatments and equal replication r, the variance of the within-block estimator of $\underline{C}'\underline{\tau}$ is a product of three quantities: the variance of the response on each plot, σ^2; the scale factor $\frac{\sum c_i^2}{r}$; and a number $\underline{C}^{'}M\underline{C}/\sum c_i^2$, where M is any generalized inverse of the scaled information matrix, $I - (nk)^{-1}NN'$, of the design: see, for instance, Bailey and Royle (1997). Hence,

$$\text{Var}\,(\underline{C}'\underline{\hat{\tau}}) = \sigma^2 \underline{C}^{'} M \underline{C}/r \qquad (1)$$

METHODS
Given the incidence matrix $N_i (i = 1, 2, 3)$ of the quotient block design of each square under consideration which is a (12×36) treatments-by-blocks matrix whose (i, j)th entry is either 0 or 1 (since each of them is a binary design) indicating the number of times that treatment i occurs in block j; the corresponding scaled information matrix is

$$L^* = I - (nk)^{-1} NN' \qquad (2)$$

where I is a conformable identity matrix. The matrix

$$L^{*+} = (L^* + aJ)^{-1},\ a \neq 0 \qquad (3)$$

where J is an all-ones matrix, is a generalized inverse of L^*: see for instance, Cameron *et al.* (2003) and Chigbu (2004). For $a = 1$, the generalized inverse matrix L^{*+}, of L^* for each square is obtainable.

We remind that $\underline{C}'\underline{\tau}$ is a contrast if $\sum c_i = 0$. Again, for an elementary contrast, $\underline{C}^{'} = (1, -1)$, and $\underline{\tau} = \binom{\tau_i}{\tau_{i'}}, i \neq i^{'}$, such that $\underline{C}'\underline{\tau} = \tau_i$-$\tau_{i'}$. Let $\underline{\hat{\tau}} = \underline{t} = \binom{t_i}{t_{i'}}$ denote the estimator of $\underline{\tau}$, then $\underline{C}^{'}\underline{\hat{\tau}} = \underline{C}^{'}\underline{t} = t_i - t_{i'}$, $i \neq i^{'}$. For each square under investigation, the number of replications $r = n$. By ignoring the constant σ^2, setting $r = n$, and $M = L^{*+}$, equation (1) is equivalent to

$$Var(\underline{C}'\underline{\hat{\tau}}) = \underline{C}^{'} L^{*+} \underline{C}\,/n \qquad (4)$$

which reduces to

$$Var(t_i - t_{i'}) = (L_{ii}{}^{*+} + L_{i'i'}{}^{*+} - L_{ii'}{}^{*+} - L_{i'i}{}^{*+})/n\,,\ i \neq i^{'} \qquad (5)$$

for elementary contrasts; where $L_{ii'}{}^{*+}$, for instance, is the $(i, i^{'})$th entry of the generalized inverse matrix, L^{*+}.

A MATLAB program was written to compute the variance of the estimate of elementary contrasts for all possible pairs of distinct treatments t_i and $t_{i'}(i < i^{'})$, in each design; as well as the average of these variances. The results were then compared for purposes of discrimination amongst the squares. The square which has a minimal number of distinct variances of these contrasts, and at the same time a minimal value of the maximum variance amongst them is considered the most preferable.

RESULTS AND DISCUSSION
The computed variances of the sixty-six (66) estimated elementary contrasts of treatments, alongside their associated frequency for the squares Λ_1, Λ_2 and Λ_3 are displayed in Tables 4, 5 and 6, respectively.

Table 4: Variances (V) of elementary treatment contrasts with their corresponding frequency (F) for Λ_1

Λ_1	**V**	0.5945	0.6072	0.6124	0.6161	0.6280	0.6667	0.6838	0.6875	0.7143	0.7195	0.7500
	F	8	8	8	4	8	4	8	8	1	8	1

Table 5: Variances (V) of elementary treatment contrasts with their corresponding frequency (F) for Λ_2

Λ_2	**V**	0.6111	0.7000	0.7111
	F	36	18	12

Table 6: Variances (V) of elementary treatment contrasts with their corresponding frequency (F) for Λ_3

Λ_3	**V**	0.6061	0.6364	0.6970
	F	30	6	30

The average of the variances of these contrasts for each square was obtained to be approximately the same in value (0.65, to two decimal places). Hence, the squares are equally optimal based on this criterion.

Furthermore, from Table 4, it is obvious that the square Λ_1 has eleven distinct values for the variance of these contrasts; while for Λ_2 and Λ_3, each of them has three distinct values of this variance, which is comparatively minimal: see Tables 5 and 6, respectively. Thus Λ_2 and Λ_3 are equally optimal and preferable to Λ_1 with respect to this criterion.

Again, in discriminating between Λ_2 and Λ_3; we observe that for Λ_3, the contrasts have a maximum variance of 0.6970, which is minimal compared to that exhibited by Λ_2 whose maximum variance of contrast is 0.7111: see Tables 6 and 5, respectively. Hence, on the basis of this criterion, Λ_3 is preferable to Λ_2.

However, we observe further that the 30 contrasts in Λ_3 whose variance is 0.6970, the maximum value for this square, are: $t_1 - t_4$, $t_1 - t_6$, $t_1 - t_8$, $t_1 - t_{10}$, $t_1 - t_{12}$, $t_2 - t_3$, $t_2 - t_5$, $t_2 - t_7$, $t_2 - t_9$, $t_2 - t_{11}$, $t_3 - t_6$, $t_3 - t_7$, $t_3 - t_9$, $t_3 - t_{12}$, $t_4 - t_5$, $t_4 - t_8$, $t_4 - t_{10}$, $t_4 - t_{11}$, $t_5 - t_8$, $t_5 - t_9$, $t_5 - t_{11}$, $t_6 - t_7$, $t_6 - t_{10}$, $t_6 - t_{12}$, $t_7 - t_{10}$, $t_7 - t_{11}$, $t_8 - t_9$, $t_8 - t_{12}$, $t_9 - t_{12}$ and $t_{10} - t_{11}$; while for Λ_2, the 12 contrasts with a variance of 0.7111 which coincide with the maximum for this square are: $t_1 - t_5$, $t_1 - t_9$, $t_2 - t_6$, $t_2 - t_{10}$, $t_3 - t_7$, $t_3 - t_{11}$, $t_4 - t_8$, $t_4 - t_{12}$, $t_5 - t_9$, $t_6 - t_{10}$, $t_7 - t_{11}$, and $t_8 - t_{12}$; and for Λ_1, the only contrast which produce the variance of 0.7500, which is the maximum for this square is $t_1 - t_8$.

CONCLUSION

We have obtained the variances of the estimate of all the 66 elementary treatment contrasts for each of the squares under consideration. Hence, they are all connected designs. The average variance of these elementary treatment contrasts for each square is approximately the same. The squares Λ_2 and Λ_3 have the same and minimal number of the aforementioned distinct pairwise treatment variances amongst these squares, which makes them preferable to Λ_1. Again, Λ_3 has a comparatively minimal value of the maximum pairwise treatment variances, thus preferable to Λ_2. Hence, on the basis of our discrimination criteria, we have found Λ_3 to be the most preferable square amongst them, for experimentation; while Λ_2 is preferable to Λ_1.

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