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NUMERICAL SOLUTIONS OF FIFTH ORDER BOUNDARY VALUE PROBLEMS USING MAMADU-NJOSEH POLYNOMIALS

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ABSTRACT

Mamadu-Njoseh polynomials are polynomials constructed in the interval [-1,1] with respect to the weight function $w(x) = x^2 + 1$. This paper aims at applying these polynomials, as trial functions satisfying the boundary conditions, in a numerical approach for the solution of fifth order boundary value problems. For this, these polynomials map the interval [0,1] to the interval [-1,1] bijectively, implying these polynomials are orthogonal in [0,1]. Numerical experiments are performed for both linear and nonlinear boundary value problems to verify the accuracy of the proposed method. Results obtained are compared with those of B-spline function method available in the literature.

Keywords: Boundary value problem, Orthogonality, Bijective mapping, Mamadu-Njoseh Polynomials, Approximate solution.

1.0. INTRODUCTION

1.1. Background

Fifth order boundary value problems are prevalent in the mathematical stimulations of Viscoelastic flow, heat convection, and in many other fields of science and technology. However, analytic methods of solving these problems are often challenging. Hence, researchers have turned their search light to numerical solution methods. In recent years, researchers take delight in solving these problems by employing polynomials as trial functions in the approximation of the exact solution. Hossain and Islam (2014) adopted the Legendre polynomials for the numerical solution of general fourth order two point boundary value problems by Galerkin method. Similarly, Olagunji and Joseph (2013) employed the third-kind Chebychev method for the numerical solutions of boundary value problems in a collocation approach. Mamadu and Njoseh (2016a) constructed orthogonal polynomials known as Mamadu-Njoseh polynomials for the numerical solution of Volterra integral equations by Galerkin method. Caglar *et al.*(1999) explored the numerical solution of fifth order boundary value problems with sixth-degree B-spline functions. Also, Rashidinia *et al.* (2011) used the non-polynomial spline solutions for special linear tenth-order boundary value problems. Other conventional methods include, the power series approximation method (PSAM) (Njoseh and Mamadu, 2016), the homotopy perturbation method (HPM) (Grover and Tomer, 2011), the differential transform method (DTM) (Islam *et al.,* 2009), Tau-collocation approximation approach (Mamadu and Njoseh, 2016b), the weighted residual method (WRM) (Oderinu, 2014), the optimal homotopy asymptotic method (OHAM) (Ali *et al.,* 2010), etc.

This paper aims at applying the Mamadu-Njoseh polynomials in Mamadu and Njoseh (2016a) as trial functions in the numerical solution of fifth order boundary value problems. For this, a bijective mapping in the interval [-1,1] to the interval [0,1] was achieved in order to modify the Mamadu-Njoseh polynomials so as to satisfy orthogonality and boundary conditions of the problems considered. The accuracy of the method is demonstrated via numerical examples with the resulting numerical results compared with those of other methods available in literature.

1.2. The Orthogonal Polynomials

Let

$$\int_a^b \mathrm{w}(\mathrm{x})\varphi_i(x)\varphi_j(x)\mathrm{dx} = \mathrm{h_i}\delta_{\mathrm{ij}} \qquad (1)$$

where δ_{ij} is the Kronecker delta denoted by

$$\delta_{\mathrm{ij}} = \begin{cases} 0, & \mathrm{i} \neq \mathrm{j} \\ 1, & \mathrm{i} = \mathrm{j} \end{cases}$$

and the weight function $w(x)$ is continuous and positive on $[a, b]$ such that the moments

$$\mu = \int_a^b \mathrm{w}(\mathrm{x})x^i\mathrm{dx}, \qquad \mathrm{i} = 0, 1, 2, 3, \ldots \qquad (2)$$

exist.

Thus the integral,

$$\langle\varphi_i, \varphi_j\rangle = \int_a^b \mathrm{w}(\mathrm{x})\varphi_i(x)\varphi_j(x)\mathrm{dx} \qquad (3)$$

is called the inner product of the polynomials φ_i and φ_j, with the orthogonality property,

$$\langle\varphi_i, \varphi_j\rangle = \int_a^b \mathrm{w}(\mathrm{x})\varphi_i(x)\varphi_j(x)\mathrm{dx} = 0, \quad i \neq j, x \in [-1,1] \qquad (4)$$

If $\delta_{\mathrm{ij}} = 1$, then the polynomials are not only orthogonal but orthonormal.

Hence we adopt the weight function $w(x) = x^2 + 1$in the interval $[a, b] \equiv [-1,1]$

The construction of φ_i, i = 1,2,3,… of the approximant:

$$\tilde{y}(x) = \sum_i^a a_i\varphi_i(x) \cong y(x) \qquad (5)$$

then follows.

1.3. Mamadu-Njoseh Polynomials

The realization of Mamadu-Njoseh polynomials (Mamadu and Njoseh, 2016a) with

$$w(x) = x^2 + 1, x \in [-1,1] \tag{6}$$

was based on these three properties;

i. $\varphi_n(x) = \sum_{i=0}^{n} C_i^{(n)} x^i$
ii. $< \varphi_m(x), \varphi_n(x) >= 0, m \neq n,$
iii. $\varphi_n(x) = 1,$

where $\varphi_i, i = 0,1,2,3,...$, are the orthogonal polynomials.

Hence, the first seven Mamadu-Njoseh polynomials as in Mamadu and Njoseh (2016a) are given in (7).

$$\left.\begin{aligned}
\varphi_0(x) &= 1\\
\varphi_1(x) &= x\\
\varphi_2(x) &= \frac{1}{3}(5x^2 - 2)\\
\varphi_3(x) &= \frac{1}{5}(14x^3 - 9x)\\
\varphi_4(x) &= \frac{1}{648}(333 - 2898x^2 + 3213x^4)\\
\varphi_5(x) &= \frac{1}{136}(325x - 1410x^3 + 1221x^5)\\
\varphi_6(x) &= \frac{1}{1064}(-460 + 8685x^2 - 24750x^4 + 17589x^6)
\end{aligned}\right\} \tag{7}$$

1.4. Modified Mamadu-Njoseh polynomials

The Mamadu-Njoseh polynomials were constructed such that they map the interval $[-1,1]$ to the interval $[0,1]$ bijectively. Hence, these polynomials are orthogonal in the interval $[0,1]$ satisfying the boundary conditions. For this, let

$$\varphi_n^*(x) = \sum_{i=0}^{n} C_i^{(n)} x^i = \left(\frac{2x-a-b}{b-a}\right) \;, \tag{8}$$

where $\varphi_i^*(x), i = 0,1,2,3,...,$ are the shifted orthogonal polynomials in the interval $[a,b]$. Thus, using (8) on the Mamadu-Njoseh polynomials in the interval $[0,1]$, we obtain the modified Mamadu-Njoseh polynomials otherwise called the **shifted Mamadu-Njoseh** polynomials as equations (9).

$$\left.\begin{aligned}
\varphi_0^*(x) &= 1\\
\varphi_1^*(x) &= 2x - 1\\
\varphi_2^*(x) &= \frac{1}{3}(20x^2 - 20x + 1)\\
\varphi_3^*(x) &= \frac{1}{5}(112x^3 - 168x^2 + 66x - 5)\\
\varphi_4^*(x) &= \frac{1}{9}(238x^4 - 1428x^3 + 910x^2 - 196x + 9)\\
\varphi_5^*(x) &= \frac{1}{17}(4884x^5 - 12210x^4 + 10800x^3 - 3990x^2 + 550x - 17)\\
\varphi_6^*(x) &= \frac{1}{133}(140712x^6 - 422136x^5 + 478170x^4 - 252780x^3 + 62010x^2 - 5976x + 333)
\end{aligned}\right\} \tag{9}$$

2.0. MATERIALS AND METHODS

2.1. Mathematical Formulation of Proposed Method

We consider the general fifth boundary value problems given by

$$\begin{gathered}
f_0(x)y^{(5)}(x) + f_1(x)y^{(4)}(x) + f_2(x)y^{(3)}(x) + \\
f_3(x)y^{(2)}(x) + f_4(x)y^{(1)}(x) + f_5(x)y(x) = r(x), \\
a < x < b
\end{gathered} \tag{10}$$

subject to the boundary conditions

$$y(0) = A_0, y'(0) = A_1, y''(0) = A_2, y(1) = B_0, y'(1) = B_1, \tag{11}$$

where $f_i(x), y^i(x), i = 0(1)5,$ and $r(x)$ are assumed real and continuous on $[0,1]$, α_i , $i = 0,1,2$ and β_i , $i = 0,1,$ are finite real constants.

Transforming (10) – (11) to systems of ordinary differential equations, we have

$$y = y_1, \frac{dy_1}{dx} = y_2, \frac{dy_2}{dx} = y_3, \frac{dy_3}{dx} = y_4, \frac{dy_4}{dx} = y_5 \tag{12}$$

subject to the boundary conditions

$$y_1(0) = A_0, \; y_2'(0) = A_1, \; y_3''(0) = A_2, \; y_4(1) = B_0, \; y_5'(0) = B_1 \tag{13}$$

Let an approximate solution of (10) – (11) be given as

$$\tilde{y}_n(x) = \sum_{i=0}^{n-1} \tau_i \varphi_i^*(x) \equiv y(x), \; x \in [0,1], \tag{14}$$

where τ_i , $i = 0(1)4$ are constants to be determined.

Substituting (14) in (11) and (12), we obtain the matrix formulation given as

$$Ax = b \tag{15}$$

Here A, x and b are 5×5, 5×1 and 5×1matrices respectively. The elements of A, x and b are a_{ij}, x_j and b_j respectively, where

$$a_{ij} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & -\frac{20}{3} & \frac{66}{5} & -\frac{196}{9} \\ 0 & 0 & \frac{40}{3} & -\frac{336}{5} & \frac{1820}{9} \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & \frac{20}{3} & \frac{66}{5} & \frac{196}{9} \end{pmatrix}, \quad x_j = \begin{pmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} \text{and} \quad b_j = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ B_0 \\ B_1 \end{pmatrix}. \tag{16}$$

Solving for the unknowns in (15) and substituting back in (14) yields the required approximate solution for the fifth order boundary value problems.

The absolute error for this formulation is given as

$$|y(x) - y_n(x)| \le \delta, \tag{17}$$

where $y(x)$ is the exact solution, $y_n(x)$ is the approximate solution and $\delta \le 10^{-6}$

2.2. Numerical Examples

Example 2.1 (Caglar et al., 1999):

Consider the following nonlinear boundary value problem of fifth order,

$$y^{(v)}(x) = e^{-x} y^2(x), \tag{18}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y''(0) = 1,\ y(1) = y'(1) = e\,. \qquad (19)$$

The exact solution is

$$y(x) = e^{x} \qquad (20)$$

Following the proposed formulation we have,

$$y(x) = 1 + x + 0.5x^2 + 0.154845484x^3 + 0.06343655x^4 \quad (21)$$

Numerical results are shown in **Table 1** for Example 2.1.

Example 2.2 (Caglar et al., 1999]):
Consider the following linear boundary value problem of fifth order,

$$y^{(v)}(x) = y - 15e^{x} - 10xe^{x}, \qquad (22)$$

subject to the boundary conditions

$$y(0) = 0,\ y'(0) = 1, y''(0) = 0,\ y(1) = 0, y'(1) = -e \qquad (23)$$

The exact solution is

$$y(x) = x(1-x)e^{x} \qquad (24)$$

Following the proposed formulation we have,

$$y(x) = x - 0.281718172x^3 - 0.718281828x^4 \qquad (25)$$

Numerical results are shown in **Table 2** for Example 2.2.

3.0. RESULTS

Table 1: Shows absolute errors obtained with the proposed method compared with B-spline function method in Caglar *et al.* (1999).

x	Exact Solution	Approximate Solution	Present Method	B-spline
0.0	1.0000000000	1.0000000000	0.0000	0.0000
0.1	1.1051709180	1.1051611890	9.7290E-06	-7.0E-04
0.2	1.2214027580	1.2213402620	6.2496E-05	-7.2E-04
0.3	1.3498588080	1.3496946620	1.6415E-04	4.1E-04
0.4	1.4918246980	1.4915340810	2.9062E-04	4.6E-04
0.5	1.6487212710	1.6483204570	4.0081E-04	4.7E-04
0.6	1.8221188000	1.8216679750	4.5082E-04	4.8E-04
0.7	2.0137527070	2.0133430670	4.0964E-04	3.9E-04
0.8	2.2255409280	2.2252644140	2.7651E-04	3.1E-04
0.9	2.4596031110	2.4595029430	1.0017E-04	1.6E-04
1.0	2.7182818280	2.7182818280	0.0000	0.0000

Table 2: Shows approximate solution and absolute error estimates obtained with the proposed method as compared with that obtained from the B-spline function method in Caglar *et al.* (1999).

x	Exact Solution	Approximate Solution	Present method absolute Error	B-spline absolute error
0.00	0.0000000000	0.0000000000	0.0000E+00	0.0000
0.01	0.0994653826	0.0996464536	1.8107E-04	8.0E-03
0.02	0.1954244413	0.1965970037	1.1726E-03	1.2E-03
0.03	0.2834703497	0.2865755266	3.1052E-03	5.0E-03
0.04	0.3580379275	0.3635820222	5.5441E-03	3.0E-03
0.05	0.4121803178	0.4198926142	7.7123E-03	8.0E-03
0.06	0.4373085120	0.4460595499	8.7510E-03	6.0E-03
0.07	0.4228880685	0.4309112001	8.0231E-03	0.0000
0.08	0.3560865485	0.3615520592	5.4655E-03	9.0E-03
0.09	0.2213642800	0.2233627453	1.9985E-03	9.0E-03
0.10	0.0000000000	0.0000000000	0.0000E+00	0.0000

4.0. DISCUSSIONS

We have explored the numerical solution of fifth order boundary problems using Mamadu-Njoseh's polynomials as trial functions. Experimenting the method, we obtained a maximum error of $9.7290E-06$, which is superior to that obtained from the B-Spline function method which has a maximum error of order 10^{-4} as shown in **Table 1** for example 2.1. Similarly, we obtained a maximum error of $1.8107E-04$, which is also superior to that obtained from the B-Spline function method which has a maximum error of order 10^{-3} as shown in **Table 2** for example 2.2. These results will converge absolutely to the exact solution with an increase in the number of Mamadu-Njoseh polynomials in the approximant. All computations were obtained using maple 18 software.

5.0. Conclusion

Here in this paper, the **Mamadu-Njoseh** polynomials have been successively implemented as trial functions to solve fifth order boundary value problems. A bijective mapping in the interval $[-1,1]$ to the interval $[0,1]$ was achieved in other to modify the **Mamadu-Njoseh** polynomials so as to satisfy orthogonality and boundary conditions of the problems considered. As illustrated in section 4 via numerical experiments for both linear and nonlinear boundary value problems, it is obvious that this new method is far more superior to the B-spline function method in Caglar *et al.* (1999). Hence, the method is accurate for the solution of higher order boundary value problems.

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