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TRANSFORMED GENERATE APPROXIMATION METHOD FOR GENERALIZED BOUNDARY VALUE PROBLEMS USING FIRST-KIND CHEBYCHEV POLYNOMIALS

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ABSTRACT
In this paper, a new numerical method, the Transformed Generate Approximation Method (TGAM) is proposed for generalized boundary value problems with first-kind Chebychev polynomials as trial functions. In this method, the trial functions are substituted into the transformed system of ordinary differential equations in order to generate systems of linear algebraic equations satisfying the boundary conditions, which on solving yield the required approximate solution. The method is structurally simple as it requires no perturbation or discretization. The method is reliable in seeking the solution of boundary value problems as numerical illustrations reveal. Results obtained were compared with the exact solution and other methods available in literature. Also, convergence analysis of the method is presented. All computations are carried out with Maple 18 software.

Keywords: Boundary value problem, Chebychev polynomials, Trial functions, Approximate solution.

1.0. INTRODUCTION

The motivation of this paper is the development of a new iterative scheme for generalized boundary value problems. For this course, we will consider the generalized boundary value problem of the form:

$$y^{(n)}(x) = g(x) - g(x)y(x), \;\; a < x < b, \quad (1)$$

subject to the boundary conditions

$$y^{(2k)}(a) = \gamma_{2k}\,, \;\; k = 0(1)(n-1)\,, \quad (2)$$

$$y^{(2k)}(b) = \upsilon_{2k}, \;\; k = 0(1)(n-1), \quad (3)$$

where $f(x)$ and $g(x)$are assumed real and differentiable on $x \in [a,b]$,γ_k ,υ_k , $k = 0(1)(n-1)$, are finite real constants and n is the order of the boundary value problem. This type of problem is a frequent occurrence in the mathematical modeling of viscoelastic flow, stability and instability convection analysis of heat transfer, and in other fields of science and technology. Hence, obtaining the solution of these problems is very essential. However, finding the solution of these problems via analytical means is quite challenging. This has prompted many researchers to search for better methods for solving these problems. Recently, Njoseh and Mamadu (2016) proposed a general resolution to this problem adopting the power series approximation method (PSAM). In like manner, Caglar*et al*. (1999) seeks the numerical solution of fifth order boundary value problem with sixth degree B-spline. . Islam *et al*. (2009) used the differential transform method (DTM) for twelfth order boundary value problem. Noor and Mohyud-Din (2008) explored the fifth order boundary value problem using the variation iteration method with He's polynomials. The method of tau and tau-collocation approximation method was adopted by Mamadu and Njoseh (2016) for the solution of first and second order ordinary differential equations. Grover and Tomer (2011) issued a fascinating new approach to evaluate twelfth order boundary value problems with homotopy perturbation method (HPM). Quite interesting to see, Yisa and Adeniyi (2012) successively developed a generalized formulation for canonical polynomials for M-Th order non-over-determined ordinary differential equations. The quest for the existence and uniqueness of the boundary value problems has been explored by Agarwal (1986). Other methods such as Galerkin method (Viswanadham and Bellem, 2015), weighted residual method (Oderinu, 2014), optimal homotopy asymptotic method (Ali *et al*., 2010), etc, are all useful methods for solving boundary value problems. Also, a collocation approach with third-kind Chebychev polynomials was adopted by Olagunju and Joseph (Olagunju and Joseph, 2013) for the solution of boundary value problems.

Major complications of some of the above mentioned methods include the following: In the PSAM, the approximate solution was a typical power series which is often complicating whenever the order of the boundary value problem is odd. In the WRM, controlling the convergence of the scheme by varying the optimal parameter may result to round-off and truncated errors, while in Olagunju and Joseph collocation method, collocating at the zeros of the third-kind Chebychev polynomials for various grid points for convergence may be time wasting. Using the Galerkin method, the septic B-splines were quasilinearized and employed as trial functions which requires more computational efforts and prone to computational errors and so on.

In this paper, a new method, Transformed Generate Approximation Method (TGAM) is formulated for a generalized case of the boundary value problem. This method employs first-kind Chebychev polynomials as trial functions in the approximation of the equations (1) – (3). In this method, the trial functions $\widetilde{y_n}(x) = \sum_{i=0}^{n-1} a_i T_i(x) \equiv y(x),\;\; x \in [a,b],$ where $T_i(x),\;\; i \geq 0$,are substituted into the transformed system of ordinary differential equations in order to generate systems of linear algebraic equations satisfying the boundary conditions, which on solving yield the required approximate solution. The method requires no discretization, linearization or quasilinearization and perturbation. Also, the method requires less computational effort and converges faster than other methods available in literature. Truncation and round-off errors are also avoided.

2.0. MATERIALS AND METHODS

2.1. Description of Transformed Generate Approximation Method

Now rewriting equations (1) – (3) as a system of ordinary differential equations we have

$$y = y_1, \frac{dy}{dx} = y_2, \frac{d^2y}{dx^2} = y_3, \frac{dy^3}{dx^3} = y_4 \cdots \frac{d^{2n-1}y}{dx^{2n-1}} = y_{2n}, \frac{dy}{dx} = g(x) - r(x)y(x), \tag{4}$$

with the boundary conditions

$$y_1(a) = \gamma_0,\ y_2(a) = \gamma_1,\ y_3(a) = \gamma_2, \cdots\ y_{2n}(a) = \gamma_{2n-1}, \tag{5}$$

$$y_1(b) = \upsilon_0,\ y_2(b) = \upsilon_1,\ y_3(b) = \upsilon_2, \ldots,\ y_{2n}(b) = \upsilon_{2n-1}. \tag{6}$$

We define an approximation solution to equations (1) – (3) as

$$y_n(x) = \sum_{i=0}^{n-1} \alpha_i T_i\ (x),\ x \in [a,b], \tag{7}$$

where:

(i) n is the order of the boundary value problem considered,

(ii) $\alpha_i,\ i = 0(1)(n-1)$, are unknown parameter to be determined,

(iii) $T_i(x),\ i \geq 0, x \in [a,b]$ are the first-kind shifted Chebychev polynomials satisfying the relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \tag{8}$$

with the initial conditions, $T_0(x) = 1$ and $T_1(x) = \frac{2x-a-b}{b-a}$,

(iv) $T_i(x),\ i \geq 0, x \in [-1,1]$ are the standard first kind-Chebychev polynomials obtained with the relations

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \tag{9}$$

with the initial conditions, $T_0(x) = 1$ and $T_1(x) = x$.
Substituting equation (7) in equations (4) - (6) with $x \in [a,b]$, we obtain the matrix equation

$$Ax = b, \tag{10}$$

where the elements of A, x and b (with elements denoted as a_{ij}, x_j and b_j) are respectively given by

$$A = \begin{bmatrix} T_0(a) & T_1(a) & T_2(a) & T_3(a) & T_4(a) & T_5(a) & T_6(a) & \cdots & T_{n-1}(a) \\ 0 & T_1'(a) & T_2'(a) & T_3'(a) & T_4'(a) & T_5'(a) & T_6'(a) & \cdots & T_{n-1}'(a) \\ 0 & 0 & T_2''(a) & T_3''(a) & T_4''(a) & T_5''(a) & T_6''(a) & \cdots & T_{n-1}''(a) \\ 0 & 0 & 0 & T_3'''(a) & T_4'''(a) & T_5'''(a) & T_6'''(a) & \cdots & T_{n-1}'''(a) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ T_0(b) & T_1(b) & T_2(b) & T_3(b) & T_4(b) & T_5(b) & T_6(b) & \cdots & T_{n-1}(b) \\ 0 & T_1'(b) & T_2'(b) & T_3'(b) & T_4'(b) & T_5'(b) & T_6'(b) & \cdots & T_{n-1}'(b) \\ 0 & 0 & T_2''(b) & T_3''(b) & T_4''(b) & T_5''(b) & T_6''(b) & \cdots & T_{n-1}''(b) \\ 0 & 0 & 0 & T_3'''(b) & T_4'''(b) & T_5'''(b) & T_6'''(b) & \cdots & T_{n-1}'''(b) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$x = [\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \cdots \quad \alpha_{n-1}]^T$$

$$b = [\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4 \quad \cdots \quad \gamma_{n-1} \quad \upsilon_0 \quad \upsilon_1 \quad \upsilon_2 \quad \upsilon_3 \quad \upsilon_4 \quad \cdots \quad \upsilon_{n-1}]^T$$

The unknown parameters $\alpha_i,\ i = 0,1,2,\ldots,(n-1)$ are computed from equation (10), and substituting these parameters into equation (7) gives the approximate to equation (1).

2.2 Convergence Analysis of the Method

Let us write the approximate solution as a power series in x

$$y(x) = \sum_{r=0}^{\infty} \propto_r x^r, x \in [a,b]. \tag{11}$$

Replacing x^r with the first kind Chebychev polynomials in Equation (11), we have

$$y(x) = \sum_{i=0}^{\infty} \alpha_i T_i(x),\ x \in [a,b]. \tag{12}$$

We then have that the series approximation (12) converges rapidly than equation (11). If a truncated series (12) is defined at $T_n(x)$, then the approximate solution of (1) – (3) is the partial sum

$$y_n(x) = \sum_{i=0}^{n-1} \alpha_i T_i\ (x), \tag{13}$$

which is a good approximation in the sense

$$\max_{a \leq x \leq b} |y(x) - y_n(x)| \leq |a_{n+1}| + |a_{n+2}| + \cdots \leq \epsilon.$$

Thus, it is possible to retain a number of terms in equation (13) for a given ϵ.

2.2. Numerical Examples

Example 2.1 (Caglar, 1999)***:***
Consider the following nonlinear boundary value of fifth order,

$$y^{(v)}(x) = e^{-x} y^2(x), \tag{14}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y''(0) = 1,\ y(1) = y'(1) = e.$$

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The exact solution is

$$y(x) = e^x.$$

From equation (10) we have,

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & -8 & 18 & -32 \\ 0 & 0 & 16 & -19 & 320 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 8 & 18 & 32 \end{pmatrix},\ x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ e \\ e \end{pmatrix}.$$

Solving the above system for the unknown parameters and substituting into (7) we then obtain the approximate solution

$$y(x) = 1 + x + 0.5x^2 + 0.154845484x^3 + 0.06343655x^4$$

Results obtained are given in Table 1.

Example 2.2 (Njoseh and Mamadu, 2016):
Consider the following thirteenth- order problem

$$y^{(13)}(x) = \cos x - \sin x, \tag{15}$$

Subject to the boundary conditions

$$y^{(0)}(0) = 1,\ y^{(1)}(0) = 1,\ y^{(2)}(0) = -1,\ y^{(3)}(0) = -1,$$
$$y^{(4)}(0) = 1,\ y^{(5)}(0) = 1,\ y^{(6)}(0) = -1,\ y^{(0)}(1) = 1,$$
$$y^{(1)}(1) = -1,\ y^{(2)}(1) = -1,\ y^{(3)}(1) = 1,\ y^{(4)}(1) = 1,$$
$$y^{(5)}(1) = -1,$$

The exact solution is

$$y(x) = \sin x + \cos x.$$

Using the proposed methodology in section 2 we have

$$y(x) = 1 + x - \tfrac{1}{2}x^2 + 0.1666666667x^3 + 0.0416666667x^4 + 0.008333333333x^5 - 0.00138888889x^6 - 53.15833333x^7 + 199.6458333x^8 - 310.7638889x^9 + 248.6791667x^{10} - 101.7416667x^{11}$$

Results obtained for different values of x are given in Table 2.

Example 2.3 (Viswanadham and Bellem, 2015):
Consider

$$y^{(10)}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, \tag{16}$$

subject to the boundary conditions

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 1,\ y'''(0) = -1,\ y^{(iv)}(0) = 1,$$
$$y(1) = e^{-1},\ y'(1) = -e^{-1},\ y''(1) = e^{-1},\ y'''(1) = -e^{-1},$$
$$y^{(iv)}(1) = e^{-1}$$

The exact solution is

$$y(x) = e^{-x}.$$

By same procedure as in section 2, we have

$$y(x) = 1 + \tfrac{1}{2}x^2 - 0.1666666667x^3 + 0.0416666667x^4 - 0.00833314x^5 + 0.0013880x^6 - 0.0001966x^7 + 0.0000228x^8 - 0.00000168x^9.$$

Results obtained are given in Table 3.

Example 2.4 (Islam et al., 2009):
Consider

$$y^{(xii)}(x) + xy(x) = -(120 + 23x + x^3)e^x, \tag{17}$$

subject to the boundary conditions

$$y(0) = 0, y'(0) = 1, y''(0) = 0,\ y'''(0) = -3,\ y^{(iv)}(0) = -8,$$
$$y^{(v)}(0) = -15\ y(1) = 0,$$

$$y'(1) = -e, y''(1) = -4e, y'''(1) = -9e,\ y^{(iv)}(1) = -16e,$$
$$y^{(iv)}(1) = -25e.$$

The exact solution is

$$y(x) = x(1 - x)\exp(x).$$

Using the proposed methodology in section 2 we have

$$y(x) = x - \tfrac{1}{2}x^3 - \tfrac{1}{3}x^4 - \tfrac{1}{8}x^5 - \frac{865}{4}x^6 + \frac{1909}{24}ex^6 + \frac{22075}{24}x^7 - \frac{2707}{8}ex^7 - \frac{3189}{2}x^8 + \frac{7039}{12}ex^8 + \frac{11241}{8}x^9 - \frac{6203}{12}ex^9 + \frac{1847}{8}ex^{10} - \frac{7531}{12}x^{10} + \frac{907}{8}x^{11} - \frac{1001}{24}ex^{11}.$$

Results obtained are given in Table 4.

3. Results

Table 1: Comparison of absolute error estimates obtained by TGAM and B-spline method in Caglar (1999)

x	Exact Solution	TGAM Solution	Absolute Error	B-spline Error
0.0	1.0000000000	1.0000000000	0.0000	0.0000
0.1	1.1051709180	1.1051611890	9.7290E-06	-7.0E-04
0.2	1.2214027580	1.2213402620	6.2496E-05	-7.2E-04
0.3	1.3498588080	1.3496946620	1.6415E-04	4.1E-04
0.4	1.4918246980	1.4915340810	2.9062E-04	4.6E-04
0.5	1.6487212710	1.6483204570	4.0081E-04	4.7E-04
0.6	1.8221188000	1.8216679750	4.5082E-04	4.8E-04
0.7	2.0137527070	2.0133430670	4.0964E-04	3.9E-04
0.8	2.2255409280	2.2252644140	2.7651E-04	3.1E-04
0.9	2.4596031110	2.4595029430	1.0017E-04	1.6E-04
1.0	2.7182818280	2.7182818280	0.0000	0.0000

Table 2: Comparison of the approximate solutions obtained by the TGAM and the exact solution

x	Exact Solution	TGAM Solution	Absolute Error
0.01	1.0099498	1.0099498	0.0000E+00
0.02	1.0197987	1.0197987	0.0000E+00
0.03	1.0295455	1.0295455	1.0000E-09
0.04	1.0391894	1.0391894	8.0000E-09
0.05	1.0487294	1.0487294	3.5000E-08
0.06	1.0581645	1.0581644	1.1800E-07
0.07	1.0674938	1.0674935	3.3500E-07
0.08	1.0767164	1.0767156	8.1900E-07
0.09	1.0858313	1.0858295	1.7950E-06
0.10	1.0948376	1.0948340	3.6060E-06

Table 3: Comparison of the approximate solutionsobtained by TGAM and the Galerkin method in Viswanadham and Bellem (2015)

x	Exact Solution	TGAM Solution	Absolute error	Error from Galerkin method
0.1	0.9048374180	0.9048374181	1.0000E-10	6.735325E-06
0.2	0.8187307531	0.8187307531	0.0000E+00	4.410744E-06
0.3	0.7408182207	0.7408182207	0.0000E+00	3.629923E-05
0.4	0.6703200460	0.6703200462	2.0000E-10	4.839897E-05
0.5	0.6065306597	0.6065306599	2.0000E-10	4.929304E-05
0.6	0.5488116361	0.5488116363	2.0000E-10	3.945827E-05
0.7	0.4965853038	0.4965853040	2.0000E-10	9.834766E-06
0.8	0.4493289641	0.4493289643	2.0000E-10	1.996756E-06
0.9	0.4065696597	0.4065696597	0.0000E+00	5.066395E-06

Table 4: The approximate solution and the absolute error estimates. The absolute error estimates are compared with that obtained by DTM in Islam *et al.* (2009)

x	Exact Solution	TGAM Solution	Absolute Error	Error from DTM
0.1	0.099465382620	0.099465382620	0	-1.6376E-15
0.2	0.195424441300	0.195424441300	0	-2.0797E-13
0.3	0.283470349700	0.283470349700	0	-3.4360E-12
0.4	0.358037927500	0.358037927500	0	-2.4556E-11
0.5	0.412180317800	0.412180317800	0	-1.1021E-10
0.6	0.437308512000	0.437308512200	2.0E-10	-3.6677E-10
0.7	0.422888068500	0.422888068500	0	-9.8945E-10
0.8	0.356086548500	0.356086548400	1.0E-10	-2.2839E-9
0.9	0.221364280000	0.221364280000	0	-4.6760E-9
1.0	0.000000000000	0.000000000000	0	-8.7157E-9

3.0. DISCUSSIONS

The proposed method has been effectively deployed to solve fifth, thirteenth, tenth and twelfth order boundary value problems. In Example 4.1, the maximum error incurred by the Transformed Generate Approximation Method is in order of 10^{-6} which is more superior to that of B-spline which is of order 10^{-4} as shown in Table 1**.** Similarly, the maximum errors obtained for Example 4.2and 4.3are of orders of 10^{-9} and 10^{-10} respectively as shown in Tables 2and **3.** Also, results obtained for Example 4.4by the TGAM is evidently far superior to that of DTM in Islam *et al.* (2009). Results obtained from all examples considered in this paper were obtained using maple 18 software.

4.0. Conclusion

From the numerical evidences obtained by the proposed method as compared with other existing methods available in literature, the transformed generate approximation method is very reliable, effective and accurate. The method requires no perturbation or discretization. Similarly, truncation and round-off errors are avoided. This method can be extended to solve linear and nonlinear integro-differentials, gas dynamic problems and hydrodynamics.

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