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CONTINUOUS MULTISTEP METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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ABSTRACT
A new class of numerical methods for Volterra integro-differential equations of the second order is developed. The methods are based on interpolation and collocation of the shifted Legendre polynomial as basis function with Trapezoidal quadrature rules. The convergence analysis revealed that the methods are consistent and zero stable, hence their convergence. Numerical examples revealed that the methods compared favourably with existing standard methods.

Keywords: Consistency, Zero stable, Continuous multistep methods, Volterra integro-differential equation, Convergent, Trapezoidal rule

INTRODUCTION

This paper discusses the numerical solution of the second order initial value problems of the Volterra type integro-differential equations of the form:

$$y''(x)= f(x, y(x), z(x)), y(x_0)= y_0, , y'(x_0)= y'_0 \quad .... (1.0)$$

where

$$z(x)= \int_{x_0}^{x} K(x,t,y(t))dt$$

METHODOLOGY

The methods of solution for second order initial value problems for ordinary differential equations of the form:

$$y''(x)= f(x, y(x)), y(x_0)= y_0, y'(x_0)= y'_0 \quad ... (2.0)$$

is considered, as discussed by Fatunla (1991), Awoyemi and Kayode (2005), Adesanya *et al*. (2009). These papers independently implemented their methods in predictor-corrector mode which is believed to have some setbacks. In order to cater for the shortcoming of the predictor-corrector methods, the block method was adopted. The block method gives solutions at each grid within the interval of integration without overlapping, and the burden of developing separate predictors is eradicated. Some earlier works on block methods include those of Jator (2007), Jator and Li (2009), D'Ambrosio *et al.* (2009), Fudziah *et al.* (2009), Yahaya and Badmus (2009), Awoyemi *et al*. (2011) and Mehrkanoon (2011), even as all of them proposed a discrete block methods that do not enable evaluation at all points within the interval of integration. In this paper we propose a continuous block method and its modification to handle second order initial value problems of the Volterra type of the form (1.0). Modifying (2.0), we can use it to solve systems of equations arising from the discretization of the second order initial value problems of the Volterra type (1.0). The idea is to approximate the exact solution $y(x)$ of (1.0) in the partition $I_n = a < x_0 < x_1 < \cdots < x_n = b$ of the integration interval $[a, b]$ with a constant step size h by the shifted Legendre polynomial of the form:

$$y(t)= \sum_{i=0}^{m} c_i p_i(t) \quad .... (2.1)$$

where $c_i \in \Re, y \in C^2(a,b)$ and $t = (x - x_n)$

The second derivative of (2.1) is substituted into (1.0) to obtain a differential system of the form

$$y'' = \sum_{i=0}^{m} c_i p''_i(t) \quad .... (2.2)$$

Now interpolating (2.1) at $x_{n+r}, r = 0$ and $k-1$ and collocating (2.2) at $x_{n+r}, r = 1, \dots, k$ and after some substitutions and manipulations, we obtain the continuous scheme of the form:

$$y(x)= \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}, y(x_{n+j}), z(x_{n+j})) \quad .... (2.3)$$

where,

$$z_n(x)= h\sum_{j=0}^{n} \alpha_j w_{nj} K(x_n, x_j, y_j), n > j, z_0 = 0 \quad .... (2.4)$$

and the weights w_{nj} depend on the choice of the quadrature rule. In this work, the Trapezoidal quadrature rule is adopted. Evaluating (2.3) for $k = 4$, we obtain the continuous linear multistep method after some substitutions and manipulations of the form:

$$y(x)= \sum_{j=0}^{3} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{4} \beta_j(x) f_{n+j} \quad .... (2.5)$$

where

$$\alpha_0 = 1 - \frac{1}{3h}t, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = \frac{1}{3h}t, , \alpha_4 = 1$$

$$\beta_0 = \frac{1}{2}t^2 - \frac{25}{72h}t^3 + \frac{35}{288h^2}t^4 - \frac{1}{48h^2}t^5 + \frac{1}{720h^4}t^6 - \frac{49}{160}ht$$

$$\beta_1 = \frac{2}{3h}t^3 + \frac{3}{40h^3}t^5 - \frac{13}{36h^2}t^4 - \frac{1}{180h^4}t^6 - \frac{39}{40}ht$$

$$\beta_2 = -\frac{1}{2h}t^3 - \frac{1}{10h^3}t^5 + \frac{19}{48h^2}t^4 + \frac{1}{120h^4}t^6 - \frac{9}{80}ht$$

$$\beta_3 = \frac{1}{9h}t^3 + \frac{1}{120h^3}t^5 - \frac{7}{36h^2}t^4 - \frac{1}{180h^4}t^6 - \frac{1}{8}ht$$

$$\beta_4 = -\frac{1}{24h}t^3 - \frac{1}{80h^3}t^5 + \frac{11}{288h^2}t^4 + \frac{1}{720h^4}t^6 + \frac{3}{160}ht \quad .... (2.6)$$

Evaluating (2.6) and its first derivative at the points $x_{n+1}, x_{n+2}, x_{n+4}$ and $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} respectively with $t = (x - x_n)$ and substituting in (2.5), we obtain the following discrete schemes:

$$y_{n+1} = \frac{2}{3}y_n - \frac{3}{5}h^2 f_{n+1} - \frac{37}{120}h^2 f_{n+2} - \frac{2}{45}h^2 f_{n+3} + \frac{1}{240}h^2 f_{n+4} + \frac{1}{3}h^2 y_{n+3} - \frac{37}{720}h^2 f_n$$

$$y_{n+2} = \frac{1}{3}y_n - \frac{7}{20}h^2 f_{n+1} - \frac{67}{120}h^2 f_{n+2} - \frac{13}{180}h^2 f_{n+3} + \frac{1}{240}h^2 f_{n+4} + \frac{2}{3}h^2 y_{n+3} - \frac{17}{720}h^2 f_n$$

$$y_{n+4} = -\frac{1}{3}y_n + \frac{11}{30}h^2 f_{n+1} - \frac{37}{360}h^2 f_{n+2} + \frac{83}{90}h^2 f_{n+3} + \frac{3}{40}h^2 f_{n+4} + \frac{4}{3}h^2 y_{n+3} + \frac{7}{360}h^2 f_n$$

$$y'_n = \frac{1}{3h}y_{n+3} - \frac{49}{160}hf_n - \frac{1}{3h}y_n - \frac{39}{40}hf_{n+1} - \frac{9}{80}hf_{n+2} - \frac{1}{8}hf_{n+3} + \frac{3}{160}hf_{n+4}$$

$$y'_{n+1} = \frac{1}{3h}y_{n+3} + \frac{61}{1440}hf_n - \frac{1}{3h}y_n - \frac{7}{90}hf_{n+1} - \frac{23}{48}hf_{n+2} + \frac{1}{45}hf_{n+3} - \frac{11}{1440}hf_{n+4}$$

$$y'_{n+2} = \frac{1}{3h}y_{n+3} + \frac{23}{1440}hf_n - \frac{1}{3h}y_n + \frac{29}{72}hf_{n+1} + \frac{37}{240}hf_{n+2} - \frac{29}{360}hf_{n+3} + \frac{11}{1440}hf_{n+4} \quad ..\ (2.7)$$

$$y'_{n+3} = \frac{1}{3h}y_{n+3} + \frac{1}{32}hf_n - \frac{1}{3h}y_n + \frac{3}{10}hf_{n+1} + \frac{63}{80}hf_{n+2} + \frac{2}{5}hf_{n+3} - \frac{3}{160}hf_{n+4}$$

$$y'_{n+4} = \frac{1}{3h}y_{n+3} + \frac{7}{1440}hf_n - \frac{1}{3h}y_n + \frac{161}{360}hf_{n+1} + \frac{101}{240}hf_{n+2} + \frac{467}{360}hf_{n+3} + \frac{95}{288}hf_{n+4}$$

Similarly, evaluating (2.3) for $k = 5$, we get

$$y(x) = \sum_{i=0}^{6} c_i p_i(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.8)$$

where

$$\alpha_0 = 1 - \frac{1}{4h}t, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = \frac{1}{4h}t, \alpha_5 = 1$$

$$\beta_0 = \frac{1}{2}t^2 - \frac{137}{360h}t^3 + \frac{5}{32h^2}t^4 - \frac{17}{480h^3}t^5 + \frac{1}{240h^4}t^6 - \frac{1}{5040h^5}t^7 - \frac{94}{315}ht$$

$$\beta_1 = \frac{5}{6h}t^3 - \frac{77}{144h^2}t^4 + \frac{71}{480h^3}t^5 - \frac{7}{360h^4}t^6 + \frac{1}{1008h^5}t^7 - \frac{356}{315}ht$$

$$\beta_2 = \frac{107}{144h^2}t^4 - \frac{59}{240h^3}t^5 + \frac{13}{360h^4}t^6 + \frac{1}{504h^5}t^7 - \frac{5}{6h}t^3 - \frac{44}{315}ht$$

$$\beta_3 = -\frac{13}{24h^2}t^4 + \frac{49}{240h^3}t^5 - \frac{1}{30h^4}t^6 + \frac{1}{504h^5}t^7 + \frac{5}{9h}t^3 - \frac{152}{315}ht$$

$$\beta_4 = \frac{61}{288h^2}t^4 - \frac{41}{480h^3}t^5 + \frac{11}{720h^4}t^6 - \frac{1}{1008h^5}t^7 - \frac{5}{24h}t^3 + \frac{4}{63}ht \quad .(2.9)$$

$$\beta_5 = -\frac{5}{144h^2}t^4 + \frac{7}{480h^3}t^5 - \frac{1}{360h^4}t^6 + \frac{1}{5040h^5}t^7 + \frac{1}{30h}t^3 - \frac{4}{315}ht$$

Evaluating (2.9) and its first derivative at the points $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+5}$ and $x_n x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$, x_{n+5} respectively with $t = (x - x_n)$ and substituting in (2.8), we obtain the following main discrete schemes

$$y_{n+1} = \frac{3}{4}y_n - \frac{337}{480}h^2 f_{n+1} + \frac{53}{120}h^2 f_{n+2} - \frac{71}{240}h^2 f_{n+3} - \frac{1}{240}h^2 f_{n+4} - \frac{1}{480}h^2 f_{n+5} + \frac{1}{4}y_{n+4} - \frac{13}{240}h^2 f_n$$

$$y_{n+2} = \frac{1}{2}y_n - \frac{8}{15}h^2 f_{n+1} - \frac{13}{15}h^2 f_{n+2} - \frac{8}{15}h^2 f_{n+3} - \frac{1}{30}h^2 f_{n+4} + \frac{1}{4}y_{n+4} - \frac{1}{30}h^2 f_n$$

$$y_{n+3} = \frac{1}{4}y_n - \frac{127}{480}h^2 f_{n+1} - \frac{29}{60}h^2 f_{n+2} - \frac{161}{240}h^2 f_{n+3} - \frac{1}{15}h^2 f_{n+4} + \frac{1}{480}h^2 f_{n+5} + \frac{3}{4}y_{n+4} - \frac{1}{60}h^2 f_n$$

$$y_{n+4} = y_n - \frac{1424}{315}h^2 f_{n+1} + \frac{176}{315}h^2 f_{n+2} + \frac{608}{315}h^2 f_{n+3} - \frac{16}{63}h^2 f_{n+4} + \frac{16}{315}h^2 f_{n+5} + 4hy'_n + \frac{376}{315}h^2 f_n$$

$$y_{n+5} = \frac{23}{96}h^2 f_{n+1} - \frac{1}{4}y_n + \frac{13}{24}h^2 f_{n+2} + \frac{11}{16}h^2 f_{n+3} + \frac{15}{16}h^2 f_{n+4} + \frac{7}{96}h^2 f_{n+5} + \frac{5}{4}y_{n+4} + \frac{1}{48}h^2 f_n$$

$$y'_{n+1} = \frac{317}{10080}hf_n - \frac{1403}{10080}hf_{n+1} - \frac{1}{4h}y_n - \frac{3497}{5040}hf_{n+2} - \frac{149}{1008}hf_{n+3} - \frac{571}{10080}hf_{n+4} + \frac{61}{10080}hf_{n+5} + \frac{1}{4h}y_{n+4}$$

$$y'_{n+2} = \frac{4}{315}hf_n + \frac{191}{630}hf_{n+1} - \frac{1}{4h}y_n + \frac{1}{63}hf_{n+2} - \frac{103}{315}hf_{n+3} - \frac{1}{315}hf_{n+4} - \frac{1}{630}hf_{n+5} + \frac{1}{4h}y_{n+4}$$

$$y'_{n+3} = \frac{41}{2016}hf_n + \frac{481}{2016}hf_{n+1} - \frac{1}{4h}y_n + \frac{2887}{5040}hf_{n+2} + \frac{1159}{5040}hf_{n+3} - \frac{683}{10080}hf_{n+4} + \frac{61}{10080}hf_{n+5} + \frac{1}{4h}y_{n+4}$$

$$y'_{n+4} = \frac{4}{315}hf_n + \frac{92}{315}hf_{n+1} - \frac{1}{4h}y_n + \frac{124}{315}hf_{n+2} + \frac{296}{315}hf_{n+3} + \frac{118}{315}hf_{n+4} - \frac{4}{315}hf_{n+5} + \frac{1}{4h}y_{n+4}$$

$$y'_{n+5} = \frac{317}{10080}hf_n + \frac{1733}{10080}hf_{n+1} - \frac{1}{4h}y_n + \frac{3671}{5040}hf_{n+2} + \frac{1943}{5040}hf_{n+3} + \frac{2753}{2016}hf_{n+4} + \frac{3197}{10080}hf_{n+5} + \frac{1}{4h}y_{n+4}$$

……………………………………………………………………….(2.10)

Analysis of the Methods

Order and error constant

Expanding the block solution of (2.7) and (2.10) in Taylor's series and collecting like terms in powers of h, we obtain the following respectively:

$$\hat{C}_0 = \hat{C}_1 = \hat{C}_2 = \ldots\hat{C}_6 = (0,0,0,0,0,0,0,0,0)^T, \hat{C}_7 = \left(\frac{107}{10080}, \frac{8}{315}, \frac{9}{224}, \frac{16}{315}, \frac{3}{160}, \frac{1}{90}, \frac{3}{160}, -\frac{8}{945}\right)^T$$

and

$$\hat{C}_0 = \hat{C}_1 = \hat{C}_2 = \ldots\hat{C}_6 = (0,0,0,0,0,0,0,0,0,0)^T, \hat{C}_7 = \left(0,0,0,0,0,0,-\frac{29}{2240}, -\frac{8}{945}, -\frac{275}{12096}\right)^T$$

Hence the block method (2.7) has order $p = (5,5,5,5,5,5,5,5)^T$ and error constants of

$$\hat{C}_7 = \left(\frac{107}{10080}, \frac{8}{315}, \frac{9}{224}, \frac{16}{315}, \frac{3}{160}, \frac{1}{90}, \frac{3}{160}, -\frac{8}{945}\right)^T$$

while the block method (2.10) has varying orders of $p = 5$ and 6 with varying error constants of

$$\hat{C}_7 = \left(-\frac{29}{2240}, -\frac{8}{945}, -\frac{275}{12096}\right)^T, \hat{C}_8 = \left(-\frac{199}{24192}, -\frac{19}{945}, -\frac{141}{4480}, -\frac{1375}{24192}, -\frac{863}{60480}, -\frac{37}{3780}\right)^T$$

Consistency

Following Lambert (1991) and Fatunla (1991), the methods are consistent since they have orders $p = 5,\ 6 > 1$

Zero stability

The blocks (2.7) and (2.10) are said to be zero stable if the roots $z_r,\ r = 1, \ldots, n$ of the first characteristic polynomial $\breve{\rho}(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfy $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$

From the block (2.7), we have

$$z^8 - z^6 = 0 \text{ and } z = (0,0,0,0,0,0,-1,1)$$

From the block (2.10), we have

$$z^{10} - z^8 = 0, z = (0,0,0,0,0,0,0,0,-1,1)$$

These show that the block methods are zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

Convergence

Following Lambert (1973) and Fatunla (1991), the methods are convergent since they are both consistent and zero stable

Numerical Illustrations

The following numerical experiments are performed with the aid of MAPLE 18 software package in order to further affirm the earlier established convergence of the methods.

Example 1

The second order linear Volterra integro differential equation

$$y''(x) - \int_0^x e^{-s} \sin xy'(s)ds + y(x) = \left(\frac{1}{2}e^{-x}\sin x - \sin x\right), 0 \le x \le 1, y(0) = -1, y'(0) = 1$$

The exact solution is $y(x) = \sin x - \cos x$ (AL-Smadi *et al.,* 2013).

The errors for $n = 100$ at different step-numbers are displayed in Tables1-2

Example 2

The second order nonlinear Volterra integro differential equation

$$y''(x) + \int_0^x (y(s))^2 ds + y(x) + \left(\frac{x}{2} - \sinh x - \frac{1}{4}\sinh 2x\right) = 0, y(0) = 0, y'(0) = 1, 0 \le x \le 1,$$

The exact solution is $y(x) = sinhx$ (AL-Smadi *et al.,* 2013).

The errors for $n = 50$ at different step-numbers are displayed in Tables 3-4 below

Table 1: Comparison of Numerical Results of Example 1 for $n = 100$

x	Exact Solution	Method(2.7)	Absolute Error	AL-Smadi *et al.,* (2013)	Absolute Error
0.16	- 0.8279090768	-0.8279090779	1.1000E-9	-0.8279086005	4.76286E-7
0.32	- 0.6346688575	-0.6346688694	1.1900E-8	- 0.6346681575	7.00003E-7
0.48	- 0.4252157473	-0.4252158013	5.4000E-8	- 0.4252150356	7.11599E-7
0.64	-0.2049003165	-0.2049004678	1.5130E-7	-0.204997512	5.65335E-7
0.80	0.0206493816	0.02064905614	3.2546E-7	0.020649707	3.25377E-7
0.96	0.2456715822	0.2456709902	5.9200E-7	0.245671644	6.14195E-8

Table 2: Comparison of Numerical Results of Example 1 for $n = 100$

x	Exact Solution	Method(2.10)	Absolute Error	AL-Smadi *et al.,* (2013)	Absolute Error
0.16	- 0.8279090768	-0.8279090768	0	-0.8279086005	4.76286E-7
0.32	- 0.6346688575	-0.6346688690	1.1500E-8	- 0.6346681575	7.00003E-7
0.48	- 0.4252157473	-0.4252157999	5.2600E-8	- 0.4252150356	7.11599E-7
0.64	-0.2049003165	-0.2049004662	1.4970E-7	-0.204997512	5.65335E-7
0.80	0.0206493816	0.0206490577	3.2387E-7	0.020649707	3.25377E-7
0.96	0.2456715822	0.2456709917	5.9050E-7	0.245671644	6.14195E-8

Table 3: Comparison of Numerical Results of Example 2 for $n = 50$

x	Exact Solution	Method (2.7)	Absolute Error	AL-Smadi *et al.,* (2013)	Absolute Error
0.16	0.1606835410	0.1606834952	4.5800E-8	0.1606828700	6.70931E-7
0.32	0.3254893636	0.3254889920	3.7160E-7	0.3254880336	1.33004E-6
0.48	0.4986455052	0.4986442191	1.2861E-6	0.4986436927	1.81246E-6
0.64	0.6845942276	0.6845910717	3.1559E-6	0.6845922814	1.92460E-6
0.80	0.8881059822	0.8880995436	6.4386E-6	0.8881044338	1.54832E-6
0.96	1.1144017940	1.1143900760	1.1718E-5	1.1144013728	4.20849E-7

Table 4: Comparison of Numerical Results of Example 2 for $n = 50$

x	Exact Solution	Method (2.10)	Absolute Error	AL-Smadi *et al.,* (2013)	Absolute Error
0.16	0.1606835410	0.1606834953	4.5700E-8	0.1606828700	6.70931E-7
0.32	0.3254893636	0.3254889921	3.7150E-7	0.3254880336	1.33004E-6
0.48	0.4986455052	0.4986442193	1.2859E-6	0.4986436927	1.81246E-6
0.64	0.6845942276	0.6845910721	3.1555E-6	0.6845922814	1.92460E-6
0.80	0.8881059822	0.8880995441	6.4381E-6	0.8881044338	1.54832E-6
0.96	1.1144017940	1.1143900760	1.1718E-5	1.1144013728	4.20849E-7

Discussion of Results

Based on the findings from this work, we have successfully established that the shifted Legendre polynomials can as well be used as basis function in the formulation of both continuous and discrete multistep method for solving the Volterra integro differential equations. The continuous formulation which was obtained via interpolation and collocation was evaluated at some selected grid points to generate the discrete block method. Unlike the discrete methods approach where additional equations are supplied from a different formulation, all our additional equations are obtained from the same continuous formulation for each step number k. The method was then applied in block form as simultaneous numerical integrators over non-overlapping intervals.

Conclusion

The performance of the newly constructed method on linear and as well as nonlinear Volterra integro-differential equations increases with increase in the step number of the methods as can be seen with the results obtained with the five step block method is better than that obtained with four step method and the accuracy of the results improve as the step sizes reduces. Generally, the new methods were found to compare favourably with the existing methods in terms of accuracy.

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