Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

Full Length Research Article

SPECIAL ALGORITHM FOR THE NUMERICAL SOLUTION OF SYSTEM OF INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS USING BLOCK HYBRID EXTENDED TRAPEZOIDAL RULE OF SECOND KIND

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ABSTRACT
We develop self-starting family of three and five step continuous extended trapezoidal rule of second kind a block hybrid type ($BHETR_2s$) methods through interpolation and collocation procedure. The $BHETR_2s$ methods are then used to produce multiple numerical integrators which are each of the same order and assembled into a single block matrix equation. These equations are simultaneously applied to provide the approximate solution for the ordinary differential equations. The stability properties of the methods were investigated and found to be consistent, zero-stable and hence convergent. The block integrators were tested on three numerical initial value problems of ODEs to show accuracy and efficiency.

Keywords: $BhETR_2s$, Zero-Stability, Convergence, General Linear Method, Collocation Method, Trapezoidal Rule, Ordinary Differential Equation

INTRODUCTION
Ordinary differential equations (ODE's) are important tools in solving real world problems and a wide variety of natural phenomena are modeled by these ODE's. Over the years, several researchers have considered the collocation method as a way of generating numerical solutions to ordinary differential equations. The collocation method is dated back as 1956 in the research carried out by Lanczos (1956) and subsequently by Brunner (1996). Lanczos (1956) introduced the standard collocation method with some selected points. However, Fox and Parker (1968) introduced the use of Chebyshev polynomials in collocating the existing method, which was captioned as the Lanczos-Tau method. Ortiz (1969) went on to discuss the general Tau method, which was later extended by Onumanyi and Ortiz (1984) to a method known as the collocation Tau Method. The standard collocation method with method of selected points provides a direct extension of the Tau method to linear ode's with non polynomial coefficients. The collocation Tau method however, uses the Chebyshev perturbation terms to select the collocation points. Tau-method was extensively studied by Onumanyi and Okunuga (1985, 1986), Okunuga and Sofoluwe (1990). Other researchers such as Fatokun (2007), Onumanyi *et al* (1994), Adeneyi and Alabi (2006), introduced some other variants of the collocation methods which recently led to some continuous collocation approach.

Collocation methods are widely considered as a way of generating numerical solution to ordinary differential equation of the form:

$$\frac{d}{dx}(y) = f(x, y),\ y(x_0) = y_0 \qquad (1)$$

This class of problems arises in the study of (1) is used in simulating the growth of populations, simple harmonic motion, trajectory of a particle etc. With the advent of modern high speed electronic digital computers, the numerical integrators have been successfully applied to study problems in mathematics, engineering, computer science and atmospheric sciences, see Jain *et al* (2007).

Many numerical integration schemes to generate the numerical solution to problem (1) have been proposed by several authors such as Butcher (2003), Awoyemi *et al* (2007), Jator (2010) and Akinfenwa (2011).

METHODOLOGY: DERIVATION OF THE METHODS
The k -step linear multistep method for the solution of (1) is given in the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} \qquad (2)$$

Which has $k+3$ unknowns α_j and $\beta_j, j = 0,1,...k$ therefore can be of order $k+2$. According to Dahlquist (1963), the order of (2) cannot exceed $k+1$ (for k odd) and $k+2$ (for k even) for the method to be stable. Authors such as Gear (1965) and Butcher (1980) have proposed modified forms of (2) which were shown to overcome the Dahlquist barrier theorem. These methods known as hybrid methods were obtained by incorporating off-step points in the derivation process.

We developed a k -step continuous hybrid formula which is an extension of (2) and involves $k(x, y)$ evaluated at off-grid points (x_{n+v}, y_{n+v}), $0 < q < k, q \notin \{0,1,..., k\}$ in the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_q f_{n+q} \qquad (3)$$

where $\alpha_k = 1$, α_0 and β_0 not both zero, $y_{n+j} = y(x + jh)$ and $f_{n+v} = f(x_{n+v}, y_{n+v})$, Lambert (1973, 1991). A method such as (3) preserves the traditional advantage of one step methods of being self-starting and permitting easy change of step length, Lambert (1973). Their advantage over R-K methods lies in the fact that they are

Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

less expensive in terms of the number of function evaluation for a given order. The methods also generate simultaneous solutions at all grid points.

The general k -step continuous extended trapezoidal rule of second kind, a hybrid type (CHETR2s) with one off-grid collocation point is given by:

$$y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h\sum_{j=v-1}^{v}\beta_j(x)f_{n+j}+h\beta_q(x)f_{n+q} \qquad (4)$$

where $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ are the continuous coefficients of the method, q is a chosen midpoint of the subinterval $\left[x_{n+k-1},x_{n+k}\right]$, Gear (1965).

From equation (4), we obtained the D and C matrix as:

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & . & . & . & x_n^{k+2} \\ 1 & x_{n+1} & x_{n+1}^2 & . & . & . & x_{n+1}^{k+2} \\ 1 & x_{n+2} & x_{n+2}^2 & . & . & . & x_{n+2}^{k+2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & . & . & . & x_{n+k-1}^{k+2} \\ 0 & 1 & 2x_{n+v-1} & . & . & . & (k+2)x_{n+v-1}^{k+1} \\ 0 & 1 & 2x_{n+v} & . & . & . & (k+2)x_{n+v}^{k+1} \\ 0 & 1 & 2x_{n+q} & . & . & . & (k+2)x_{n+q}^{k+1} \end{bmatrix} \qquad (5)$$

and

$$C=\begin{bmatrix} \alpha_{v-2,1} & \alpha_{v-1,1} & . & . & . & \alpha_{v,1} & h\beta_{m-1,1} & . & . & . & h\beta_{m,1} & h\beta_{q,1} \\ \alpha_{v-2,2} & \alpha_{v-1,2} & . & . & . & \alpha_{v,2} & h\beta_{m-1,2} & . & . & . & h\beta_{m,2} & h\beta_{q,2} \\ . & . & & & & . & . & & & & . & . \\ . & . & & & & . & . & & & & . & . \\ . & . & & & & . & . & & & & . & . \\ \alpha_{v-2,k+2} & \alpha_{v-1,k+2} & & & & \alpha_{v,k+2} & h\beta_{m-1,k+2} & & & & h\beta_{m,k+2} & h\beta_{q,k+2} \end{bmatrix} \qquad (6)$$

Equations (4), (5) and (6) are therefore used to derived the continuous formulation of the hybrid ETR2s with one off-grid collocation point for step numbers $k=3$ and 5.

2.1: Consider $k=3,\ q=\frac{5}{2}$, equation (4) becomes:

$$y(x)=\sum_{j=0}^{2}\alpha_j(x)y_{n+j}+h\sum_{j=1}^{2}\beta_j(x)f_{n+j}+h\beta_{\frac{5}{2}}(x)f_{n+\frac{5}{2}} \qquad (7)$$

From (7), we obtained (5) and (6) as:

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \end{bmatrix} \qquad (8)$$

and

$$C=\begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \alpha_{2,1} & h\beta_{1,1} & h\beta_{u,1} & h\beta_{2,1} \\ \alpha_{0,2} & \alpha_{1,2} & \alpha_{2,2} & h\beta_{1,2} & h\beta_{u,2} & h\beta_{2,2} \\ \alpha_{0,3} & \alpha_{1,3} & \alpha_{2,3} & h\beta_{1,3} & h\beta_{u,3} & h\beta_{2,3} \\ \alpha_{0,4} & \alpha_{1,4} & \alpha_{2,4} & h\beta_{1,4} & h\beta_{u,4} & h\beta_{2,4} \\ \alpha_{0,5} & \alpha_{1,5} & \alpha_{2,5} & h\beta_{1,5} & h\beta_{u,5} & h\beta_{2,5} \\ \alpha_{0,6} & \alpha_{1,6} & \alpha_{2,6} & h\beta_{1,6} & h\beta_{u,6} & h\beta_{2,6} \end{bmatrix} \qquad (9)$$

After some computations, the values of $\alpha_0(x),\ \alpha_1(x),\ \alpha_2(x),\ \beta_1(x),\ \beta_{\frac{5}{2}}(x)$ and $\beta_2(x)$ are obtained from (8) and (9) and substituted into equation (7) to give us the desired continuous formulation for the three step block hybrid ETR2s in the form:

$$\begin{aligned} y(x_n+\xi) :=& \left(-\frac{145}{43}\frac{\xi}{h}+\frac{751}{172}\frac{\xi^2}{h^2}-\frac{233}{86}\frac{\xi^3}{h^3} +\frac{139}{172}\frac{\xi^4}{h^4}+1-\frac{4}{43}\frac{\xi^5}{h^5}\right)y_n+\Big(\\ & -\frac{692}{43}\frac{\xi^3}{h^3}+\frac{652}{43}\frac{\xi^2}{h^2}-\frac{160}{43}\frac{\xi}{h}-\frac{40}{43}\frac{\xi^5}{h^5} +\frac{283}{43}\frac{\xi^4}{h^4}\Big)y_{n+1}+\left(\frac{1617}{86}\frac{\xi^3}{h^3}+\frac{305}{43}\frac{\xi}{h}\right. \\ & \left.+\frac{44}{43}\frac{\xi^5}{h^5}-\frac{3359}{172}\frac{\xi^2}{h^2}-\frac{1271}{172}\frac{\xi^4}{h^4}\right)y_{n+2} +\left(\frac{195}{43}\frac{\xi^4}{h^3}+\frac{648}{43}\frac{\xi^2}{h}-\frac{1633}{129}\frac{\xi^3}{h^2}\right. \\ & \left.-\frac{820}{129}\xi-\frac{76}{129}\frac{\xi^5}{h^4}\right)f_{n+1}+\left(-\frac{64}{43}\frac{\xi^2}{h} -\frac{32}{43}\frac{\xi^4}{h^3}+\frac{16}{129}\frac{\xi^5}{h^4}+\frac{64}{129}\xi\right. \\ & \left.+\frac{208}{129}\frac{\xi^3}{h^2}\right)f_{n+\frac{5}{2}}+\left(\frac{887}{86}\frac{\xi^2}{h}+\frac{379}{86}\frac{\xi^4}{h^3} -\frac{28}{43}\frac{\xi^5}{h^4}-\frac{155}{43}\xi-\frac{450}{43}\frac{\xi^3}{h^2}\right)f_{n+2} \end{aligned} \qquad (10)$$

Equation (10) is evaluated at some ξ points to obtained discrete equations which we called (11). These equations are combined to form a block, which are implemented to give simultaneously approximate solutions to problem (1).

$$\frac{1}{4}(43y_{n+3}-141y_{n+2}+93y_{n+1}+5y_n) =\frac{h}{2}\left[32f_{n+\frac{5}{2}}-39f_{n+2}-23f_{n+1}\right]$$

$\frac{1}{60}(2752y_{n+\frac{5}{2}} - 2025y_{n+2} - 700y_{n+1} - 27y_n) = \frac{h}{2}\left[16f_{n+\frac{5}{2}} + 45f_{n+2} + 10f_{n+1}\right]$

$\frac{1}{2}(1797y_{n+2} - 1704y_{n+1} - 93y_n) = \frac{h}{2}\left[129f_{n+3} - 640f_{n+\frac{5}{2}} + 1554f_{n+2} + 847f_{n+1}\right]$

$\frac{1}{2}(-915y_{n+2} + 480y_{n+1} + 435y_n) = \frac{h}{2}\left[64f_{n+\frac{5}{2}} - 465f_{n+2} - 820f_{n+1} - 129f_n\right]$ (11)

2.2: For $k = 5$, taking $q = \frac{9}{2}$, we generate (4) in the form:

$$y(x) = \sum_{j=0}^{4}\alpha_j(x)y_{n+j} + h\sum_{j=1}^{3}\beta_j(x)f_{n+j} + h\beta_{\frac{9}{2}}(x)f_{n+\frac{9}{2}} \quad (12)$$

In this case, our D and C matrix becomes:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+\frac{9}{2}} & 3x_{n+\frac{9}{2}}^2 & 4x_{n+\frac{9}{2}}^3 & 5x_{n+\frac{9}{2}}^4 & 6x_{n+\frac{9}{2}}^5 & 7x_{n+\frac{9}{2}}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \end{bmatrix} \quad (13)$$

and

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \alpha_{2,1} & \alpha_{3,1} & \alpha_{4,1} & h\beta_{2,1} & h\beta_{u,1} & h\beta_{3,1} \\ \alpha_{0,2} & \alpha_{1,2} & \alpha_{2,2} & \alpha_{3,2} & \alpha_{4,2} & h\beta_{2,2} & h\beta_{u,2} & h\beta_{3,2} \\ \alpha_{0,3} & \alpha_{1,3} & \alpha_{2,3} & \alpha_{3,3} & \alpha_{4,3} & h\beta_{2,3} & h\beta_{u,3} & h\beta_{3,3} \\ \alpha_{0,4} & \alpha_{1,4} & \alpha_{2,4} & \alpha_{3,4} & \alpha_{4,4} & h\beta_{2,4} & h\beta_{u,4} & h\beta_{3,4} \\ \alpha_{0,5} & \alpha_{1,5} & \alpha_{2,5} & \alpha_{3,5} & \alpha_{4,5} & h\beta_{2,5} & h\beta_{u,5} & h\beta_{3,5} \\ \alpha_{0,6} & \alpha_{1,6} & \alpha_{2,6} & \alpha_{3,6} & \alpha_{4,6} & h\beta_{2,6} & h\beta_{u,6} & h\beta_{3,6} \\ \alpha_{0,7} & \alpha_{1,7} & \alpha_{2,7} & \alpha_{3,7} & \alpha_{4,7} & h\beta_{2,7} & h\beta_{u,7} & h\beta_{3,7} \\ \alpha_{0,8} & \alpha_{1,8} & \alpha_{2,8} & \alpha_{3,8} & \alpha_{4,8} & h\beta_{2,8} & h\beta_{u,8} & h\beta_{3,8} \end{bmatrix} \quad (14)$$

The unknown coefficients of the method (12) are obtained after solving D and C matrix where $C = D^{-1}$, then substituted back into (12) to yield the CHETR2s for the five step block hybrid ETR2s in the form:

$\frac{1}{24}(2193y_{n+5} + 4545y_{n+4} + 16640y_{n+3} - 21120y_{n+2} - 2385y_{n+1} + 127y_n = \frac{h}{2}\left[256f_{n+9/2} + 1555f_{n+3} - 1059f_{n+2}\right]$

$\frac{1}{2520}(748544y_{n+9/2} - 1488375y_{n+4} - 661500y_{n+3} + 1285956y_{n+2} + 121500y_{n+1} - 6125y_n) = \frac{h}{2}\left[128f_{n+9/2} - 1050f_{n+3} - 567f_{n+2}\right]$

$\frac{1}{24}(-166095y_{n+4} - 376540y_{n+3} + 490320y_{n+2} + 55260y_{n+1} - 2945y_n) = \frac{h}{2}\left[645f_{n+5} - 2816f_{n+9/2} - 36455f_{n+3} - 24549f_{n+2}\right]$

$\frac{1}{24}(-321735y_{n+4} - 108880y_{n+3} + 397980y_{n+2} + 34320y_{n+1} - 1685y_n) = \frac{h}{2}\left[1024f_{n+9/2} - 10965f_{n+4} - 37640f_{n+3} - 17694f_{n+2}\right]$

$\frac{1}{24}(48405y_{n+4} + 499100y_{n+3} - 196560y_{n+2} - 342300y_{n+1} - 8645y_n) = \frac{h}{2}\left[256f_{n+9/2} + 23485f_{n+3} + 44919f_{n+2} + 10965f_{n+1}\right]$

$\frac{1}{8}(60885y_{n+4} + 509520y_{n+3} - 393660y_{n+2} - 222480y_{n+1} + 45735y_n) = \frac{h}{2}\left[1024f_{n+9/2} + 79320f_{n+3} + 113886f_{n+2} - 3655f_{n+1}\right]$ (16)

Stability Analysis

In this paper, we shall discuss the following terminology; order and error constant, consistency and zero-stability of linear multistep methods.

3.1: Order and error constant

Following Fatunla (1991) and Lambert (1973), the local truncation error associated with equation (3) is the linear difference operator $\mathcal{L}$ as

$$\mathcal{L}\{y(x),h\} = \sum_{j=0}^{k}\left\{\alpha_j y(x_{n+j}) - h\beta_j y(x_{n+j})\right\} \quad (17)$$

The Taylor series expansion of (17) about the point x_n yield

$$\mathcal{L}\{y(x),h\} = c_0 y(x_n) + c_1 h y'(x_n) + \ldots + c_q h^q y^{(q)}(x_n) + \ldots \quad (18)$$

where,

$$c_0 = \sum_{j=0}^{k}\alpha_j \quad (19)$$

$$c_1 = \sum_{j=0}^{k} j\alpha_j \quad (20)$$

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.
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$$c_q = \frac{1}{q!}\sum_{j=0}^{k} j\alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{(q-1)}\alpha_j \quad (21)$$

where,

c_q are the constant coefficients

According to Henrici (1962), the method (3) has order p if $c_0 = c_1 = ... = c_p = 0$ and $c_{p+1} \neq 0,$ where c_{p+1} is called the error constant.

3.2: Consistency
A Linear multistep method (3) is consistent if it has order greater or equal to one (that is $p \geq 1$) that is, if

(i) $\rho(1) = 0$

(ii) $\rho'(1) = \sigma(1)$

where, ρ and σ are the first and second characteristic polynomials of the method.

3.3: Zero-Stability
A Linear multistep method (3) is said to be zero stable if no roots of the first characteristic polynomial $\rho(\xi)$ has modulus greater than one and every root with modulus one is distinct, Lambert (1973, 1991).

3.4: General Linear Method (GLM)
The GLM were introduced by Burrage and Butcher (1980) and Shirley (2005) on the implementation of LMM for the numerical solution of equation (1).
The GLM for the solution of problem (1) can be expressed in the form

$$Y_i^{[n+1]} = h\sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j^{[n+1]}) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, i = 1,..., s \qquad (22)$$

$$Y_i^{[n+1]} = h\sum_{j=1}^{s} b_{ij} f(x_n + c_i h, Y_j^{[n+1]}) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, i = 1,..., r \qquad (23)$$

$n > 0, n = 0,1,..., N$, $h = (x_{n+i} - x_n)/N$ and $x_n = x_0 + nh$ and $i = 1,2,..., s$ are stage values which are internal to each step and represent approximations to the solution at the point $x_n + c_i h$, the vector of external values $y^{[n]} = [y_1^{[n]}, y_2^{[n]},..., y_r^{[n]}]^T$ denotes information available at the end of the n^{th}-step for the input to the next step. Again we emphasize that r denotes quantities as output from each step and input to the next step and s denotes step values used in the computation.
These methods are characterized by four matrices A, U, B and V which make up a partition $(s+r)\times(s+r)$ matrix M in the form:

$$M = \begin{bmatrix} A & . & U \\ . & . & . \\ B & . & V \end{bmatrix} \qquad (24)$$

and the general linear method takes the form

$$\begin{bmatrix} Y^{[n]} \\ \cdots \\ y^{[n]} \end{bmatrix} = M \begin{bmatrix} hf(Y^{[n]}) \\ \cdots \\ y^{[n-1]} \end{bmatrix} \qquad (25)$$

where,

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ . \\ . \\ . \\ Y_s^{[n]} \end{bmatrix}, \; f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ . \\ . \\ . \\ f(Y_s^{[n]}) \end{bmatrix},$$

$$y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ . \\ . \\ . \\ y_r^{[n-1]} \end{bmatrix} \quad \text{and} \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ . \\ . \\ . \\ y_r^{[n]} \end{bmatrix}$$

From (22) to (25), the following definitions hold

Definition 1:
For a GLM (25), the stability matrix $M(z)$ is given by

$$M(z) = V + zB(1 - zA)^{-1}U \qquad (26)$$

and the characteristic polynomial given by

$$\rho(\lambda, z) = \det(\lambda I - M(z)) \qquad (27)$$

Butcher (2003).

Definition 2:
A General Linear Method is A-stable if for all $z \in C^{-1}, I - zA$ is non-singular and $M(z)$ is a stability matrix, Butcher (2003).

Note:
i. For an A-stable method, there is no restriction on the choice of the step size.
ii. A-stability helps us to determine the type of problems our methods can handle.

Analysis of our Newly Derived Methods
4.1: Three Step Block Hybrid ETR$_2$s (BHETR$_2$s 1)
4.1.1 Order, Consistency and Convergence (BHETR$_2$s 1)

Applying (17) to (21), we obtained the order and error constant of our method (11) as $p = 5$ and $c_6 = (\frac{7}{5!}, -\frac{1}{(2!)^6}, -\frac{269}{5!}, -\frac{23}{4!})^T$ respectively. Since (11) has order $p = 5 \geq 1$, it is consistent. See (3.1)

Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

4.1.2 Zero-Stability (BHETR2s 1)

The block method (11) can be expressed in the form

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+\frac{5}{2}}\\y_{n+3}\end{pmatrix}=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n-\frac{5}{2}}\\y_{n-2}\\y_{n-1}\\y_n\end{pmatrix}$$

$$+h\left\{\begin{pmatrix}\frac{361}{360}&-\frac{101}{120}&\frac{152}{225}&-\frac{61}{360}\\\frac{64}{45}&\frac{1}{15}&\frac{64}{225}&-\frac{4}{45}\\\frac{1625}{1152}&\frac{125}{384}&\frac{5}{9}&-\frac{125}{1152}\\\frac{57}{40}&\frac{9}{40}&\frac{24}{25}&\frac{3}{40}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+\frac{5}{2}}\\f_{n+3}\end{pmatrix}+\begin{pmatrix}0&0&0&\frac{599}{1800}\\0&0&0&\frac{71}{225}\\0&0&0&\frac{365}{1152}\\0&0&0&\frac{63}{200}\end{pmatrix}\begin{pmatrix}f_{n-\frac{5}{2}}\\f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}\right\} \quad (28)$$

Hence the characteristic polynomial of the block method (11) is given by

$$\rho(\lambda)=\det\left(\lambda A^{(0)}-A^{(1)}\right)$$

$$\rho(\lambda)=\det\left(\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right)$$

$$\rho(\lambda)=\lambda^3(\lambda-1)=0 \quad \Rightarrow \lambda_1=1, \lambda_2=\lambda_3=\lambda_4=0$$

From (3.3), the method is zero-stable.

4.1.3 GLM for BHETR2s 1

We converted the three step block hybrid ETR2s (11) into GLM (24) as

$$\left[\begin{array}{ccccc|ccc}0&0&0&0&0&0&0&1\\-\frac{129}{480}&-\frac{820}{480}&-\frac{465}{480}&\frac{64}{480}&0&\frac{915}{480}&0&-\frac{435}{480}\\0&\frac{847}{1797}&\frac{1554}{1797}&-\frac{640}{1797}&\frac{129}{1797}&0&\frac{1704}{1797}&\frac{93}{1797}\\0&\frac{300}{2752}&\frac{1350}{2752}&\frac{480}{2752}&0&\frac{2025}{2752}&\frac{175}{2752}&\frac{27}{2752}\\0&-\frac{46}{43}&-\frac{78}{43}&\frac{64}{43}&0&\frac{141}{43}&-\frac{93}{43}&-\frac{5}{43}\\\hline0&-\frac{46}{43}&-\frac{78}{43}&\frac{64}{43}&0&\frac{141}{43}&-\frac{93}{43}&-\frac{5}{43}\\0&\frac{847}{1797}&\frac{1554}{1797}&-\frac{640}{1797}&\frac{129}{1797}&0&\frac{1704}{1797}&\frac{93}{1797}\\-\frac{129}{480}&-\frac{820}{480}&-\frac{465}{480}&\frac{64}{480}&0&\frac{915}{480}&0&-\frac{435}{480}\end{array}\right] \quad (29)$$

We plotted the absolute stability region of (29) as

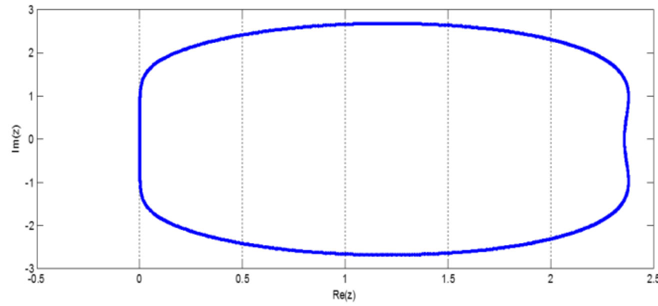

Fig. 1: Absolute stability region for BHETR2s 1 method

4.2: Five Step Block Hybrid ETR2s (BHETR2s 2)

4.2.1 Order, Consistency and Convergence (BHETR2s 2)

We obtained the order and error constant of THBH 2 (16) using (17) to (21) as $p=7$ and

$$c_8=\left(\frac{13}{7.(2!)^2},-\frac{45}{(2!)^8},-\frac{1217}{7.(2!)^4},-\frac{523}{7.(2!)^4},-\frac{97}{(2!)^4},-\frac{3447}{7.(2!)^4}\right)^T$$

respectively. Since (16) has order $p=7\geq1$, it is consistent. See (3.1)

4.2.2 Zero-Stability (BHETR2s 2)

$$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+\frac{9}{2}}\\y_{n+5}\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n-\frac{9}{2}}\\y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{pmatrix}$$

$$+h\left\{\begin{pmatrix}\frac{78553}{70560}&-\frac{4519}{5040}&\frac{13691}{15120}&-\frac{9841}{10080}&\frac{13808}{19845}&-\frac{15373}{10080}\\\frac{6691}{4410}&-\frac{5}{63}&\frac{517}{945}&-\frac{206}{315}&\frac{9472}{19845}&-\frac{67}{630}\\\frac{11601}{7840}&\frac{45}{112}&\frac{689}{560}&-\frac{1017}{1120}&\frac{464}{735}&-\frac{153}{1120}\\\frac{3296}{2205}&\frac{104}{315}&\frac{1664}{945}&-\frac{62}{315}&\frac{8192}{19845}&-\frac{32}{315}\\\frac{374463}{250880}&\frac{1215}{3584}&\frac{31077}{17920}&\frac{3159}{35840}&\frac{162}{245}&-\frac{4131}{35840}\\\frac{21125}{14112}&\frac{325}{1008}&\frac{5375}{3024}&-\frac{125}{2016}&\frac{4400}{3969}&\frac{115}{2016}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+\frac{9}{2}}\\f_{n+5}\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&\frac{28199}{725760}\\0&0&0&0&0&\frac{169}{4536}\\0&0&0&0&0&\frac{1013}{26880}\\0&0&0&0&0&\frac{85}{2268}\\0&0&0&0&0&\frac{2151}{57344}\\0&0&0&0&0&\frac{5435}{145152}\end{pmatrix}\begin{pmatrix}f_{n-\frac{9}{2}}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}\right\}$$

From (29), the first characteristic polynomial $\rho(\lambda)$ of THBH 2 (16) is in the form $\rho(\lambda)=\det(\lambda A^{(0)}-A^{(1)})=\lambda^5(\lambda-1)=0$, which implies that $\lambda_1=1$ and $\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6=0$. Hence, it is zero stable. We have therefore shown the convergence of the method.

Science World Journal Vol 12(No 4) 2017
www.scienceworldjournal.org
ISSN 1597-6343

4.2.3: GLM for BHETR$_{2}$s 2

Equation (16) is converted into GLM (24) as

$$
\left[\begin{array}{ccccccc|ccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{14620}{222480} & 0 & -\frac{455544}{222480} & -\frac{317280}{222480} & 0 & -\frac{4096}{222480} & 0 & \frac{60885}{222480} & \frac{509520}{222480} & -\frac{393660}{222480} & 0 & \frac{45735}{222480} \\
0 & -\frac{131580}{196560} & -\frac{539028}{196560} & -\frac{281820}{196560} & 0 & -\frac{3072}{196560} & 0 & \frac{48405}{196560} & \frac{499100}{196560} & 0 & -\frac{342300}{196560} & -\frac{8645}{196560} \\
0 & 0 & \frac{212328}{108880} & \frac{451680}{108880} & \frac{131580}{108880} & -\frac{12288}{108880} & 0 & -\frac{321735}{108880} & 0 & \frac{397980}{108880} & \frac{34320}{108880} & -\frac{1685}{108880} \\
0 & 0 & \frac{294588}{166095} & \frac{437460}{166095} & 0 & \frac{33792}{166095} & -\frac{7740}{166095} & 0 & -\frac{376540}{166095} & \frac{490320}{166095} & \frac{55260}{166095} & -\frac{2945}{166095} \\
0 & 0 & -\frac{714420}{748544} & -\frac{1323000}{748544} & 0 & \frac{161280}{748544} & 0 & \frac{1488375}{748544} & \frac{661500}{748544} & -\frac{1285156}{748544} & -\frac{121500}{748544} & \frac{6125}{748544} \\
0 & 0 & \frac{4236}{731} & \frac{6220}{731} & 0 & \frac{1024}{731} & 0 & -\frac{1515}{731} & -\frac{16640}{2193} & \frac{7040}{731} & \frac{795}{731} & -\frac{127}{2193} \\
\hline
0 & 0 & \frac{4236}{731} & \frac{6220}{731} & 0 & \frac{1024}{731} & 0 & -\frac{1515}{731} & -\frac{16640}{2193} & \frac{7040}{731} & \frac{795}{731} & -\frac{127}{2193} \\
0 & 0 & \frac{294588}{166095} & \frac{437460}{166095} & 0 & \frac{33792}{166095} & -\frac{7740}{166095} & 0 & -\frac{376540}{166095} & \frac{490320}{166095} & \frac{55260}{166095} & -\frac{2945}{166095} \\
0 & 0 & \frac{212328}{108880} & \frac{451680}{108880} & \frac{131580}{108880} & -\frac{12288}{108880} & 0 & -\frac{321735}{108880} & 0 & \frac{397980}{108880} & \frac{34320}{108880} & -\frac{1685}{108880} \\
0 & -\frac{131580}{196560} & -\frac{539028}{196560} & -\frac{281820}{196560} & 0 & -\frac{3072}{196560} & 0 & \frac{48405}{196560} & \frac{499100}{196560} & 0 & -\frac{342300}{196560} & -\frac{8645}{196560} \\
\frac{14620}{222480} & 0 & -\frac{455544}{222480} & -\frac{317280}{222480} & 0 & -\frac{4096}{222480} & 0 & \frac{60885}{222480} & \frac{509520}{222480} & -\frac{393660}{222480} & 0 & \frac{45735}{222480}
\end{array}\right] \quad (30)
$$

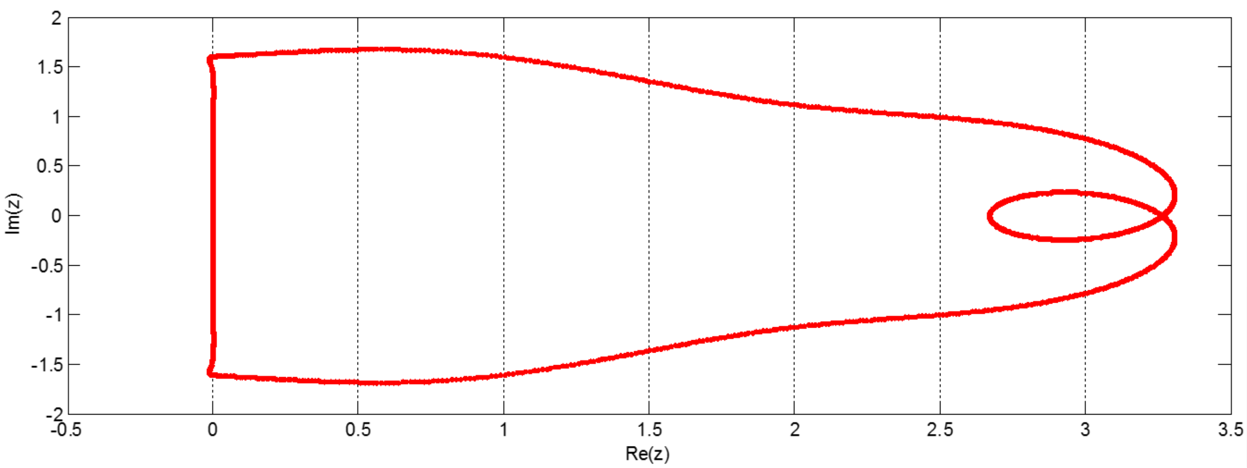

Fig. 2: Absolute stability region for **BHETR$_{2}$s 2** method

Numerical Example

In this paper, we constructed block hybrid extended trapezoidal rule of second kind (BHETR2s 1 and BHETR2s 2). Both methods are applied on three numerical problems of ordinary differential equations occurring in real life. For the sake of reporting, the hybrid block methods for step numbers $k = 3$ *and* 5 were presented. We present the performance of the newly derived block methods using the well-known MatLab ode solver, Ode23s. For the avoidance of doubt, our method(s) tends to perform very well on the MatLab Ode solver, (see Fig.3-8).

Problem 1: System of Ordinary Differential Equations in 8 dimensions

$y_1'(x) = -1.71y_1(x) + 0.43y_2(x) + 8.32y_3(x) + 0.00007y_4(x)$
$y_2'(x) = 1.71y_1(x) - 8.75y_2(x)$
$y_3'(x) = -10.03y_3(x) + 0.43y_4(x) + 0.035y_5(x)$
$y_4'(x) = 8.32y_2(x) + 0.171y_3(x) - 1.12y_4(x)$
$y_5'(x) = -1.145y_5(x) + 0.43y_6(x) + 0.43y_7(x)$
$y_6'(x) = -280y_6(x)y_8(x) + 0.69y_4(x) + 1.71y_5(x) - 0.43y_6(x) + 0.69y_7(x)$
$y_7'(x) = 280y_6(x)y_8(x) - 1.81y_7(x)$
$y_8'(x) = -280y_6(x)y_8(x) + 1.81y_7(x)$
with initial condition given by $y_1(0) = 1,\ y_2(0) = 0,\ y_3(0) = 0,\ y_4(0) = 0,\ y_5(0) = 0,\ y_6(0) = 0,\ y_7(0) = 0, y_8(0) = 0.0057, 0 \le x \le 10, h = 0.01$

Problem 2: Oregenator Chemical Reaction Equation

The Oregenator is expressed mathematically by the following *ivp*

$y_1' = 77.27(y_2 + y_1(1 - 8.375 \times 10^{-6}y_1 - y_2)), y_1(0) = 1$
$y_2' = \frac{1}{77.27}(y_3 - ((1 + y_1)y_2)\,, \qquad y_2(0) = 0$
$y_3' = 0.16(y_1 - y_3)\,, \qquad y_3(0) = -1$
$0 \le x \le 700,\ h = 0.01$

[NB: A famous chemical reaction is the Oregenator reaction between Br $HBrO_2, Br^-$ and $Ce(IV)$ described by Field and Noyes in 1984].

Problem 3: Stiff System of Ordinary Differential Equations

Consider the linear stiff system of ordinary differential equations on the range $0 \le x \le 10$.

$$y_1' = -10y_1 + 50y_2$$
$$y_2' = -50y_1 - 10y_2$$
$$y_3' = -40y_3 - 200y_4$$
$$y_4' = -200y_3 - 40y_4$$
$$y_5' = -0.2y_5 - 2y_6$$
$$y_6' = -2y_5 - 0.2y_6$$
$$y_1(0) = 0,\ y_2(0) = 1,\ y_3(0) = 0,\ y_4(0) = 1,\ y_5(0) = 0,\ y_6(0) = 1, h = 0.01$$

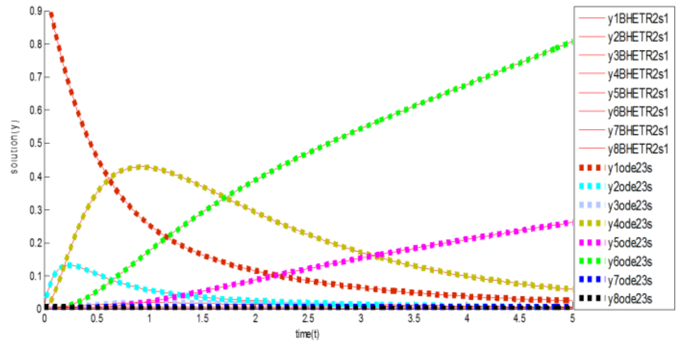

Fig.3: solution curve for problem 1 using BHETR2s 1

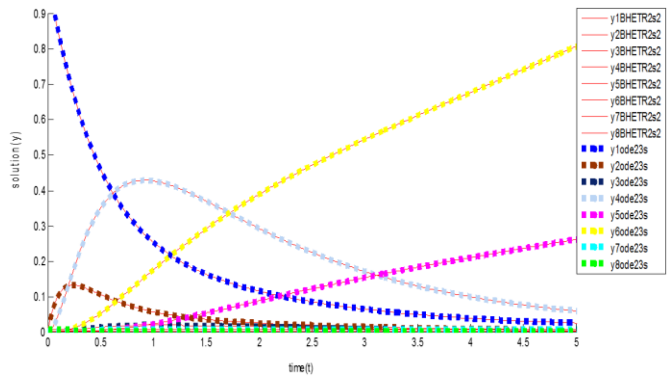

Fig 4: solution curve for problem 1 using BHETR2s 2.

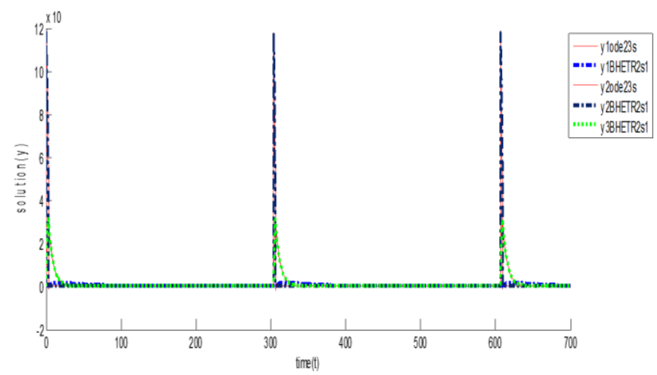

Fig. 5: solution curve for problem 2 using BHETR2s 1.

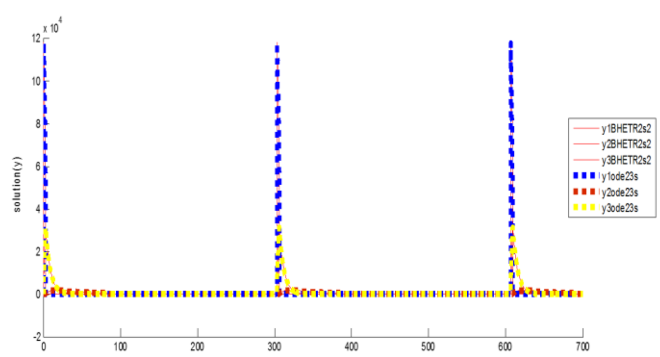

Fig. 6: solution curve for problem 2 using BHETR2s 2.

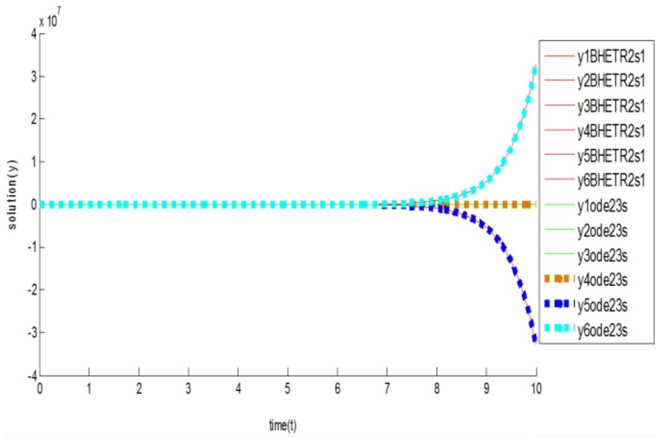

Fig. 7: solution curve for problem 3 using $BHETR_2s$ 1.

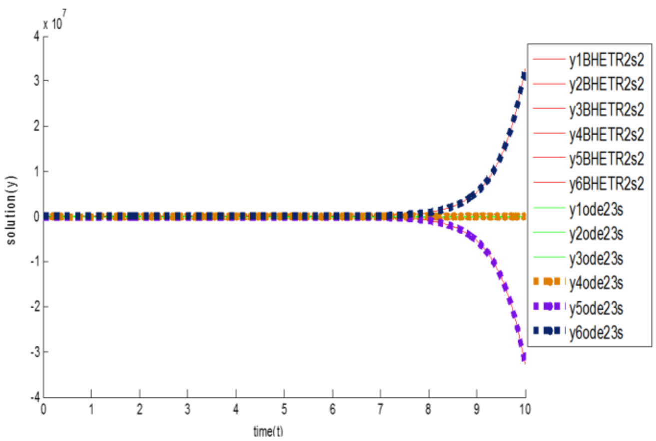

Fig. 8: solution curve for problem 3 using $BHETR_2s$ 2.

Conclusion

Our new method(s) competes quite well especially when compared with the Ode Solver. The solution curves show the performance block hybrid methods for step numbers $k = 3\ and\ 5$ was quite remarkable especially when compared with the well known $ode\ 23s$ and $ode\ 45$. The block hybrid methods were shown to be consistent, zero-stable and hence convergent (see table). Moreover, they were also shown to have order $p = k + 2$. Fig.2-8 reveal that our block hybrid methods are A-stable since their region(s) of absolute stability contain the whole of the left hand complex half plane and as such are suitable for the solution of stiff problems.

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