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EXTENSION OF FORMULAS FOR PARTITION FUNCTIONS

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ABSTRACT

This paper studies the elementary method for finding formulas for some partition functions. The idea dates back to Cayley and Macmahon, using partial fractions decomposition to obtain traceable power series expansions. The ideas are hereby extended with the aid of software called *maple package*.

Keywords: Formula, Integer Partitions, Partition functions

INTRODUCTION

Partition of a non-negative integer n is a non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda n$, that sum to n. For example the integer 4 has the following five partitions, $\{4, 31, 22, 211, 1111\}$, see Andrews (1984).

We denote by $p_k(n)$ the number of partitions of n into at most k parts, Gupta (1970).

Andrew and Erikson (2004) also stated that $p_k(n)$ equals the number of partitions of n into at most k parts. There is an extensive literature concerning the formula for $p_k(n)$, including contributions by Cayley (1856), Sylvester (1882), Glaisher (1909) and Gupta (1958). For additional references and historical notes, see Gupta (1970). Furthermore, the theory of q-partial fractions and its formula was developed in Munagi (2007).

Some Important Definitions

Partition

Singh *et al.* (2012) defined a partition of a positive integer n as a sequence of positive integers whose sum is n.

The order of the summands is unimportant when writing the partitions of n. For example the partitions of $n = 4$ are given as

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

Partition function $p(n)$

Andrew and Erikson (2004) stated that the partition function $p(n)$ counts the number of unique partitions of the positive integer n.

For example, there are five unique partitions of 4. Hence, $p(4) = 5$.

Intermediate function

The intermediate function denoted by $p_k(n)$ is defined such that it counts the partitions of n with the largest added being no smaller than k, Gupta (1970).

METHODOLOGY

In this section, we consider the formula $p_k(n)$, which was computed for $k = 1, 2, 3, 4.$ by Andrews (2003). So our desire in this paper is to compute this formula for $k = 5, 6, \ldots, 11.$ Furthermore, we implement this formula using computer program called *maple software*.

That is, Andrews (2003) deduced

$$p_1(n) = 1,$$
$$p_2(n) = \left\lfloor \frac{n+1}{2} \right\rfloor,$$
$$p_3(n) = \{(n+3)^2/12\}$$

Thereafter, Andrew and Erikson (2004) gave $p_k(n) = p(n,k)$ the following generating function

$$\sum_{n=0}^{\infty} p(n,k)q^n = \frac{1}{(1-q)(1-q^2)\ldots(1-q^k)} \qquad (1)$$

Now, consider the case $k = 4$, we have maple calculate that

$$\begin{aligned} \sum_{n=0}^{\infty} p(n,4)\, q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)} \\ &= \frac{\frac{25}{144}}{(1-q)^2} + \frac{\frac{1}{8}}{288(1-q)^3} + \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{1}{16}}{(1-q)^2} + \frac{\frac{1}{8}}{(1-q^2)(1-q)^2} + \frac{1}{q(q+2)}\left(\frac{1}{1-q^3}\right) + \frac{\frac{1}{4}}{(1-q^4)} \\ &= \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{1}{8}}{(1-q)^3} + \frac{\left(\frac{5}{12}\right)^2}{(1-q)^2} + \frac{\frac{1}{8}}{(1-q^2)^2} + \frac{\frac{1}{16}}{1-q^2} + \frac{\frac{(2+q)}{9}}{1-q^3} + \frac{\frac{1}{4}}{1-q^4} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{24}\binom{n+3}{3} + \frac{1}{8}\binom{n+2}{2} + \left(\frac{5}{12}\right)^2 (n+1)\right)q^n + \left(\frac{1}{8}(n+1) + \frac{1}{16}\right)q^{2n} \\ &\quad + \sum_{n=0}^{\infty}\left(-\frac{1}{16}q^{2n} + \frac{2}{9}q^{3n} + \frac{1}{9}q^{3n+1} + \frac{1}{4}q^{4n}\right) \end{aligned}$$

Here, the coefficient of q^n is the formula for

$$p(n,4) = \frac{1}{24}\binom{n+3}{3} + \frac{1}{8}\binom{n+2}{2} + \frac{25}{144}(n+1) \qquad (2)$$

Case $k = 5$

$$\sum_{n=0}^{\infty} p(n,5)\, q^n = 1(1-q)(1-q^2)\dots(1-q^5)\ \frac{\frac{1}{120}}{(1-q)^5} + \frac{\frac{1}{24}}{(1-q)^4} + \frac{\frac{31}{288}}{(1-q)^3} + \frac{\frac{11}{64}}{(1-q)^2} + \frac{\frac{11}{64}}{1-q^2} + \frac{\frac{1}{9}}{1-q^3} + \frac{\frac{1}{5}}{1-q^5}$$

$$= \left(\frac{1}{120}\binom{n+4}{4} + \frac{1}{24}\binom{n+3}{3} + \frac{31}{288}\binom{n+2}{2} + \frac{11}{64}\binom{n+1}{1}\right) q^n + \frac{11}{64}\sum_{n=0}^{\infty} q^{2n} + \frac{1}{9} q^{3n} + \frac{1}{5}\sum_{n=0}^{\infty} q^{5n}$$

$$p(n,5) = \frac{1}{120}\binom{n+4}{4} + \frac{1}{24}\binom{n+3}{3} + \frac{31}{288}\binom{n+2}{2} + \frac{11}{64}\binom{n+1}{1} \qquad (3)$$

Case $k = 6$

$$\sum_{n=0}^{\infty} p(n,6)\, q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^6)}$$
$$= \frac{1}{720(1-q)^6} + \frac{1}{96(1-q)^5} + \frac{17}{432(1-q)^4} + \frac{331}{3456}\left(\frac{1}{(1-q)^3}\right) + \frac{777611}{518400}\left(\frac{1}{(1-q)^2}\right) + \frac{25}{256(1-q^2)}$$
$$= \frac{1}{720}\sum_{n=0}^{\infty}\binom{n+5}{5} q^n + \frac{1}{96}\sum_{n=0}^{\infty}\binom{n+4}{4} + \frac{17}{432}\sum_{n=0}^{\infty}\binom{n+3}{3} + \frac{331}{3456}\sum_{n=0}^{\infty}\binom{n+2}{2} + \frac{777611}{518400}\sum_{n=0}^{\infty}(n+1) + \frac{25}{256} q^{2n}$$

$$p(n,6) = \frac{1}{720}\binom{n+5}{5} + \frac{1}{96}\binom{n+4}{4} + \frac{17}{432}\binom{n+3}{3} + \frac{331}{3456}\binom{n+2}{2} + \frac{777611}{518400}\binom{n+1}{1} \qquad (4)$$

Case $k = 7$

$$\sum_{n=0}^{\infty} p(n,7)\, q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^7)}$$
$$= \frac{1}{5040}\left(\frac{1}{(1-q)^6}\right) + \frac{1}{480(1-q)^5} + \frac{47}{4320}\left(\frac{1}{(1-q)^4}\right) + \frac{161}{4320}\left(\frac{1}{(1-q)^3}\right) + \frac{7913}{86400}\left(\frac{1}{(1-q)^2}\right)$$
$$= \frac{1}{5040}\sum_{n=0}^{\infty}\binom{n+5}{5} q^n + \frac{1}{400}\sum_{n=0}^{\infty}\binom{n+4}{4} q^n + \frac{47}{4320}\sum_{n=0}^{\infty}\binom{n+3}{3} q^n + \frac{161}{4320}\sum_{n=0}^{\infty}\binom{n+2}{2} q^n + \frac{7713}{86400}\sum_{n=0}^{\infty}\binom{n+1}{1} q^n$$

$$p(n,7) = \frac{1}{5040}\binom{n+5}{5} + \frac{1}{480}\binom{n+4}{4} + \frac{47}{4320}\binom{n+3}{3} + \frac{161}{4320}\binom{n+2}{2} + \frac{7913}{86400}\binom{n+1}{1} \qquad (5)$$

Case $k = 8$

$$\sum_{n=0}^{\infty} p(n,8)\, q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^8)}$$
$$= \sum_{n=0}^{\infty}\left\{\left(\frac{1}{40320}\right)\left(\frac{1}{(1-q)^8}\right) + \left(\frac{1}{2880}\right)\left(\frac{1}{(1-q)^7}\right) + \left(\frac{83}{34560}\right)\left(\frac{1}{(1-q)^6}\right) + \left(\frac{25}{2304}\right)\left(\frac{1}{(1-q)^5}\right) + \left(\frac{24523}{691200}\right)\left(\frac{1}{(1-q)^4}\right) + \left(\frac{139}{1600}\right)\left(\frac{1}{(1-q)^3}\right) + \frac{487033}{3175200}\binom{n+1}{1}\right\}$$

$$p(n,8) = \frac{1}{40320}\binom{n+7}{7} + \frac{1}{2880}\binom{n+6}{6} + \frac{83}{34560}\binom{n+5}{5} + \frac{25}{2304}\binom{n+4}{4} + \frac{24523}{691200}\binom{n+3}{3} + \frac{139}{1600}\binom{n+2}{2} + \frac{487033}{3175200}\binom{n+1}{1} \qquad (6)$$

Case $k = 9$

$$\sum_{n=0}^{\infty} p(n,9)\, q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^9)} = \sum_{n=0}^{\infty}\left\{\frac{1}{326880(1-q)^9} + \frac{1}{20160(1-q)^8} + \frac{319}{725760(1-q)^7} + \frac{1843}{725760(1-q)^6} + \frac{929377}{87091200(1-q)^5} + \frac{239185}{6967296(1-q)^4} + \frac{8335199}{97542144(1-q)^3} + \frac{1821389}{11943936(1-q)^2}\right\} q^n =$$

$$\frac{1}{326880}\binom{n+8}{8}+\frac{1}{20160}\binom{n+7}{7}+\frac{319}{725760}\binom{n+6}{6}+\frac{1843}{725760}\binom{n+5}{5}+\frac{929377}{87091200}\binom{n+4}{4}+\frac{239185}{6967296}\binom{n+3}{3}+\frac{8335199}{97542144}\binom{n+2}{2}+\frac{1821389}{11943936}\binom{n+1}{1} \quad (7)$$

Case $k = 10$

$$\sum_{n=0}^{\infty} p(n,10)\,q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^{10})}$$
$$=\sum\left\{\frac{1}{362880(1-q)^{10}}+\frac{1}{161280(1-q)^9}+\frac{199}{2903040(1-q)^8}+\frac{2867}{5806080(1-q)^7}+\frac{161111}{62208000(1-q)^6}+\frac{2600533}{248832000(1-q)^5}+\frac{1826428579}{54867456000(1-q)^4}+\frac{36618675691}{438939648000(1-q)^3}+\frac{40769204821}{268738560000(1-q)^2}\right\}q^n$$

$$p(n,10)=\frac{1}{326880}\binom{n+9}{9}+\frac{1}{161280}\binom{n+8}{8}+\frac{199}{2903040}\binom{n+7}{7}+\frac{2867}{5806080}\binom{n+6}{6}+\frac{161111}{62208000}\binom{n+5}{5}+\frac{2600533}{248832000}\binom{n+4}{4}+\frac{1826428579}{54867456000}\binom{n+3}{3}+\frac{36618675691}{438939648000}\binom{n+2}{2}+\frac{40769204821}{268738560000}\binom{n+1}{1} \quad (8)$$

Case $k = 11$

$$\sum_{n=0}^{\infty} p(n,11)\,q^n = \frac{1}{(1-q)(1-q^2)\dots(1-q^{11})}$$
$$=\sum_{n=0}^{\infty}\left\{\frac{1}{39916800(1-q)^{11}}+\frac{1}{14151520(1-q)^{10}}+\frac{1}{107520(1-q)^9}+\frac{95}{1161216(1-q)^8}+\frac{114221}{217728000(1-q)^7}+\frac{2262473}{870912000(1-q)^6}+\frac{2250121541}{219469824000(1-q)^5}+\frac{14347821041}{438939648000(1-q)^4}+\frac{1099442945011}{13168189440000(1-q)^3}+\frac{17080331207}{107495424000(1-q)^2}\right\}q^n$$

$$p(n,11)=\frac{1}{39916800}\binom{n+10}{10}+\frac{1}{1415120}\binom{n+9}{9}+\frac{1}{107520}\binom{n+8}{8}+\frac{95}{1161216}\binom{n+7}{7}+\frac{114221}{217728000}\binom{n+6}{6}+\frac{2262473}{870912000}\binom{n+5}{5}+\frac{2250121541}{219469824000}\binom{n+4}{4}+\frac{14347821041}{438939648000}\binom{n+3}{3}+\frac{1099442945011}{13168189440000}\binom{n+2}{2}+\frac{17080331207}{107495424000}\binom{n+1}{1} \quad (9)$$

Example

We illustrate the above formulas for the case n = 50, m = 10
Substituting n = 50

$$p\,(50,10)=\frac{1}{3628800}\binom{59}{9}+\frac{1}{161280}\binom{58}{8}+\frac{199}{2903040}\binom{57}{7}+\frac{2867}{5806080}\binom{56}{6}+\frac{161111}{62208000}\binom{55}{5}+\frac{2600533}{248832000}\binom{54}{4}+\frac{1826428579}{54867456000}\binom{53}{3}+\frac{36618675691}{438939648000}\binom{52}{2}+\frac{40769204821}{268738560000}\binom{51}{1}$$
$$= 16928$$

This result agrees with the one obtained with maple via inbuilt command " numbpart " from the combination package:

>with (combinat, numbpart);

> $NumbPart(50, 10);$ 16928

Conclusion

We have studied the elementary method for finding formulas for some partition functions, which used partial fractions decomposition to obtain traceable power series expansions. The ideas had been extended with the aid of Maple software package.

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APPENDIX

Maple Code for Generating q-Partial Fraction and Iterative Transformation Algorithm

```
with(numtheory, cyclotomic, divisors, phi, invphi)
local a, t, i, h;
type/qfactors := proc(f, q::name)
a := primpart (f, q)*(1 + q^7)^2;
qseries:=table();
qseries[aqprod]:=proc(a,q,n)
: added else bit when n not an integer
localx,i:
if type(n,nonnegint) then
x:=1:
fori from 1 to n do
x := x * (1-a*q^(i-1)):
od:
else
x:=``(a,q)[n];
fi:
   RETURN(x):
end:
```

>Partitions Function:

```
pmn(3,n);
pmn:=proc(m, n) local a, gu, Gu, H, gu1, lu1, M, q, i, eqs, form,
unknowns, result, soln,j,k,l;
option remember;
if m = 1 then RETURN([[1]]) end if;
Gu :=mul(1/(1-q^i), i = 1 .. m);
gu := convert(Gu, parfrac);
H := [];
fori to m do
M := floor((m-i)/i);
gu1 := add(coeff(gu, numtheory:-cyclotomic(i, q),
 -j)/numtheory:-cyclotomic(i, q)^j, j = 1 .. floor(m/i));
lu1 := convert(taylor(gu1, q = 0, i*(1+M)), polynom);
unknowns := {seq(a[j], j = 0 .. -1+i*(1+M))};
eqs := {seq(seq(add((k*i+l)^j*a[j*i+l], j = 0 .. M)
    = coeff(lu1, q, k*i+l), k = 0 .. M), l = 0 .. i-1)};
form := [seq(add(a[j*i+l]*(1-q)^j, j = 0 .. M), l = 0 .. i-1)];
soln := solve(eqs, unknowns);
result := subs(soln, form);
result:=[op(2..nops(result),result),result[1]]:
H := [op(H), result]
od:
 H:
end:
```

>Polynomials of the form $\frac{1}{\left(1-q^{n}\right)}$

```
with(numtheory, cyclotomic, divisors, ø, invphi)
type/qfactors := proc(f, q::name)
local a, t, i, h;
option
description   "Tests if f is a constant multiple of factors of polynomials of the form 1-q^n";
if not type(f, polynom(anything, q ) ) then return false end if;
a := primpart (f, q)*(1 + q^7)^2;
if op(1,a) = -1 then a:= -a end if;
t :=0;
fori to nops(a) do
if type (op(i, a), { '^', 'l' } ) then
h :=op(1, op(i,a)^2); if abs(h) = abs(1 – q^degree (h, q))then t := t
+ 1 end if
else return false
```