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Full Length Research Article

THE NON-COMMUTATIVE FULL RHOTRIX RING AND ITS SUBRINGS

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ABSTRACT

This paper presents the triple $(R_n(F),+,\circ)$ consisting of the set of all rhotrices of size *n* with entries in an arbitrary ring *F*; and together with the operations of rhotrix addition '+' and row-column based method for rhotrix multiplication; ' $\circ$ ' , so as to introduce it as *"the concept of non-commutative full rhotrix ring"*, and study its properties. A number of subrings of $(R_n(F),+,\circ)$ are uncovered. Next, the paper shows that a particular subring of the non-commutative full rhotrix ring $(R_n(F),+,\circ)$ is embedded in a particular subring of the well-known non-commutative full matrix ring $(M_n(F),+,\cdot)$. Furthermore, isomorphic relationships between some subrings of $(R_n(F),+,\circ)$ are investigated.

Keywords: Ring, subring, Rhotrix Ring, Rhotrix Subring, Matrix Ring, Matrix Subring

INTRODUCTION

A rhotrix is a rhomboidal form of representing array of numbers. A rhotrix set is a set consisting of well-defined rhotrices as its elements. A rhotrix ring is a ring having rhotrix set as an underlying set. A non-empty subset S of a ring R is called a subring of R if S is a ring under the operations of addition and multiplication defined on R. Worthy of note to say that, S is a *subring* of R if and only if $a, b \in S$ implies $a-b \in S$ and $ab \in S$ as established in Garrett (2008).

The concept of rhotrix of size *3* was introduced by Ajibade (2003) as an extension of ideas on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon (1998). Ajibade defined a collection of all rhotrices of size 3 with entries from set of all real numbers as

$$R_3(\Re)=\left\{\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix}\right\rangle : a,b,c,d,e\in\Re\right\}$$

The entry at the particular intersection of the vertical and horizontal diagonal denoted by $h(A_3)=c$ is called the heart of any rhotrix $A_3\in R_3(\Re)$. The following are the binary operations of addition (+) and multiplication (o) defined in [1], recorded respectively, as follows:

$$R_3+Q_3=\left\langle \begin{matrix} & a & \\ b & h(R_3) & d \\ & e & \end{matrix}\right\rangle+\left\langle \begin{matrix} & f & \\ g & h(Q_3) & j \\ & k & \end{matrix}\right\rangle=\left\langle \begin{matrix} & a+f & \\ b+g & h(R_3)+h(Q_3) & d+j \\ & e+k & \end{matrix}\right\rangle,$$

$$R_3\circ Q_3=\left\langle \begin{matrix} & ah(Q_3)+fh(R_3) & \\ bh(Q_3)+gh(R_3) & h(R_3)h(Q_3) & dh(Q_3)+jh(R_3) \\ & eh(Q_3)+kh(R_3) & \end{matrix}\right\rangle \quad (1)$$

$$R_n=\langle a_{ij},c_{lk}\rangle=\langle A_{t\times t},C_{(t-1)\times(t-1)}\rangle=$$

$$\left\langle \begin{matrix} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \dots & \dots\dots & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{matrix}\right\rangle,$$

where a_{ij} and c_{lk} are major and minor entries respectively. Implying that

$$R_n=\left\langle \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(t-1)} & a_{1t} \\ a_{21} & a_{22} & \dots & a_{2(t-1)} & a_{2t} \\ \dots & \dots & \dots & \dots & \dots \\ a_{(t-1)1} & a_{(t-1)2} & \dots & a_{(t-1)(t-1)} & a_{(t-1)t} \\ a_{t1} & a_{t2} & \dots & a_{t(t-1)} & a_{tt} \end{bmatrix}, \begin{bmatrix} c_{11} & \dots & c_{1(t-1)} \\ \dots & \dots & \dots \\ c_{(t-1)1} & \dots & c_{(t-1)(t-1)} \end{bmatrix}\right\rangle.$$

The set of all such collections of rhotrices with entries from an arbitrary field F can be denoted as

$$R_n(F)=\{\langle a_{ij},c_{lk}\rangle : a_{ij}\in F, c_{lk}\in F\},$$

where $1 \le i,\ j \le t,$

$$1 \le l, k \le t-1;\ t = \frac{n+1}{2}\ and\ n \in 2Z^{+}+1.$$

Multiplication of two rhotrices R_n and Q_n was defined in Sani (2007) as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle \quad (2)$$

It was noted in Sani (2007) that the row-column based rhotrix multiplication (2) is non-commutative, but associative. The identity element of any rhotrix of size n was also given as

$$I_n = \langle I_{t\times t}, I_{(t-1)\times(t-1)} \rangle = \left\langle \begin{matrix} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & \ldots & \ldots & 1 & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{matrix} \right\rangle \quad (3)$$

The determinant of a rhotrix R_n was also defined as:

$$\det(R_n) = \det\langle a_{ij}, c_{lk} \rangle = \det(A_{t\times t}).\det(C_{(t-1)\times(t-1)}).$$

It was also presented that $R_n = \langle a_{ij}, c_{lk} \rangle$ is invertible if and only if the two matrices $[a_{ij}]$ and $[c_{lk}]$ are both invertible matrices. This means that if R_n is invertible and $R_n^{-1} = \langle q_{ij}, r_{lk} \rangle$ then q_{ij} and r_{lk} are the inverse entries of matrices $A_{t\times t}$ and $C_{(t-1)\times(t-1)}$ respectively. Also, noteworthy to mention that R_n is invertible if and only if $\det(R_n) \ne 0.$ It was also shown that

$$\det(R_n \circ Q_n) = \det(R_n) \circ \det(Q_n) = \det(R_n).\det(Q_n).$$

If $R_n = \langle a_{ij}, c_{lk} \rangle$ then its transpose was defined as $R_n^T = \langle a_{ji}, c_{kl} \rangle$. It was noted that $(R_n \circ Q_n)^T = (Q_n)^T \circ (R_n)^T.$

Since the concept of rhotrix was introduced by Ajibade in 2003, some algebraic structures having rhotrix set as an underlying set had been studied in the literature of rhotrix theory. Worthy of note are Mohammed (2007a, 2007b), Tudunkaya and Makanjuola (2012a, 2012b), Aminu (2009), Mohammed and Balarabe (2014, 2017a, 2017b), Mohammed *et al.*(2014), Mohammed and Okon (2016), Mohammed and Shettima (2017a) and Mohammed and Shettima (2017b).. However, (Mohammed and Balarabe, 2014) observed in their ten year reviewed of rhotrix theory from 2003 to 2012 that, authors that adopted the rhotrix multiplication (1) in presentation of their works contributed their articles to the literature of commutative rhotrix theory. While, authors adopting rhotrix multiplication (2) for presentation of their works, contributed their articles to the literature of non-commutative rhotrix theory. Now, considering the articles in the literatures of rhotrices, the concept of non-commutative ring of rhotrices has never been discussed, to the best of our knowledge.

In this paper, an algebraic study of non-commutative ring of rhotrices is introduced. This is achieved through a consideration of the triple $(R_n(F), +, \circ)$, consisting of the set $R_n(F)$ of all rhotrices of size *n* having entries in an arbitrary ring *F* and together with the operations of rhotrix addition $(+)$ and row-column based method for rhotrix multiplication $(\circ)$, which forms the ring of all rhotrices of size *n* over an arbitrary ring *F*. The ring is termed as '*the non-commutative full rhotrix ring'*, because of the non-commutativity of row-column based method for rhotrix multiplication. The properties of the ring will be investigated. Furthermore, subrings of $(R_n(F), +, \circ)$ will be identified and it will also be shown that, a particular subring of $(R_n(F), +, \circ)$ is embedded in a particular subring of the well-known full matrix ring $(M_n(F), +, \cdot)$. In the process, a number of results are shown.

The Non-Commutative Full Rhotrix Ring

This section studies the set of all rhotrices of size *n* with entries from any arbitrary ring F and together with operations of rhotrix addition and row-column based method for rhotrix multiplication, so as to introduce it as "*the concept of non–commutative full rhotrix ring*" as given by the theorem below.

Theorem (The non-commutative full rhotrix ring)

Let $R_n(F)$ be the set of all rhotrices of size n having entries from an arbitrary ring *F*. Let $+$ and $\circ$ be the operations of rhotrix addition and row-column based multiplication of rhotrices of the same size respectively. Then, the triple $(R_n(F), +, \circ)$ is a non-commutative *full rhotrix ring* of size *n* over an arbitrary ring F.

Proof

The triple $(R_n(F), +, \circ)$ is a ring under the binary operations of addition and row-column based multiplication of all rhotrices of size *n*. That is, $(R_n(F), +, \circ)$ satisfies all the ring axioms as below.

(i) $(R_n(F), +)$ is an Abelian group

Let $R_n = \langle a_{ij}, c_{lk} \rangle$ and $S_n = \langle b_{ij}, d_{lk} \rangle$ be any two rhotrices in $R_n(F)$ .Observe that

$$R_n + S_n = \langle a_{ij}, c_{lk} \rangle + \langle b_{ij}, d_{lk} \rangle = \langle a_{ij} + b_{ij}, c_{lk} + d_{lk} \rangle \in R_n(F)$$

Thus, $R_n(F)$ is closed under the operation of rhotrix addition. Associativity holds in $(R_n(F), +)$, since for any three rhotrices $R_n, S_n, T_n \in R_n(F)$ with $T_n = \langle e_{ij}, g_{lk} \rangle$, it is easy to see that

$(R_n + S_n) + T_n = R_n + (S_n + T_n)$. Also, for any rhotrix $R_n \in R_n(F)$, there exists an identity element $O_n = \langle 0_{ij}, 0_{lk} \rangle \in R_n(F)$ such that

$$R_n + O_n = \left\langle a_{ij}, c_{lk} \right\rangle + \left\langle 0_{ij}, 0_{lk} \right\rangle =$$
$$\left\langle a_{ij} + 0_{ij}, c_{lk} + 0_{lk} \right\rangle = \left\langle a_{ij}, c_{lk} \right\rangle = R_n \in R_n(F)$$

and $O_n + R_n = R_n \in R_n(F)$ Finally, inverses elements exist, since for each rhotrix $R_n \in R_n(F)$ there exists an element $-R_n = \langle -a_{ij}, -c_{lk} \rangle \in R_n(F)$ such that

$$R_n + (-R_n) = \left\langle a_{ij}, c_{lk} \right\rangle + \left\langle -a_{ij}, -c_{lk} \right\rangle =$$
$$\left\langle a_{ij} - a_{ij}, c_{lk} - c_{lk} \right\rangle = \left\langle 0_{ij}, 0_{lk} \right\rangle = O_n \in R_n(F)$$

and $-R_n + R_n = O_n \in R_n(F)$. Also,

$$R_n + S_n = \left\langle a_{ij} + b_{ij}, c_{lk} + d_{lk} \right\rangle =$$
$$\left\langle b_{ij} + a_{ij}, d_{lk} + c_{lk} \right\rangle = S_n + R_n \in R_n(F).$$

Thus, the pair $R_n(F), +)$ is a commutative rhotrix group.

(ii) $(R_n(F), \circ)$ is a semigroup

For any two rhotrices $R_n, S_n \in R_n(F)$, we have

$$R_n \circ S_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle =$$
$$\left\langle \sum_{i_2 j_1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle \in R_n(F)$$

Thus, $(R_n(F), \circ)$ is closed under the binary operation of row-column based rhotrix multiplication. Associativity holds in $(R_n(F), \circ)$, since for all $R_n, S_n \; and \; T_n \in R_n(F)$, it can be obtained by computation that $(R_n \circ S_n) \circ T_n = R_n \circ (S_n \circ T_n)$. Hence, the pair $(R_n(F), \circ)$ is a non-commutative rhotrix semigroup.

(iii) $(R_n(F), +, \circ)$ Possesses multiplication that is left and right distributive over addition

For any three rhotrices $R_n, S_n, T_n \in R_n(F)$, it can be obtained by computation that $R_n \circ (S_n + T_n) = R_n \circ S_n + R_n \circ T_n$ and $(S_n + T_n) \circ R_n = S_n \circ R_n + T_n \circ R_n$. So multiplication is both left and right distributive over addition in $(R_n(F), +, \circ)$.

Hence, the triple $(R_n(F), +, \circ)$ is a non-commutative full rhotrix ring. Furthermore, any other set of rhotrices of the same size with entries in F that forms a non-commutative ring of rhotrices is a subset of $(R_n(F), +, \circ)$.

Corollary
Let the ring F in the above theorem be the ring $\mathbb{Z}$ of all integer numbers. Then, the triple$(R_n(\mathbb{Z}), +, \circ)$ is *the non-commutative ring of all integer rhotrices of size n.*

Proof
Putting $\mathrm{F} = \mathbb{Z}$ in the above theorem, the result follows.

Corollary
Let the arbitrary ring F in the above theorem be the ring $\mathbb{R}$ of all real numbers. Then, the triple $(R_n(\mathbb{R}), +, \circ)$ is *the non-commutative ring of all real rhotrices of size n*.

Proof
Putting $\mathrm{F} = \mathbb{R}$ in the above theorem, the result follows.

Corollary
Let $n = 3$ and letF be the ring of all complex numbers $\mathbb{C}$ in the above theorem. Then the triple $(R_3(\mathbb{C}), +, \circ)$is the non-commutative ring of all complex rhotrices of size 3.

Proof
Putting $n = 3$and $\mathrm{F} = \mathbb{C}$ in the above theorem, the result follows.

Before the next theorem, it may be of interest to recall that, Garrett (2008) noted that, the collection $M_n(F)$ of all n-by-n matrices with entries in a ring F is a non-commutative ring, with the usual matrix addition$(+)$ and multiplication$(\cdot)$. Now, we denote this full matrix ring as the triple $(M_n(F), +, \cdot)$.

Theorem
Let F be a ring. Then the non-commutative full rhotrix ring $(R_n(F), +, \circ)$ is embedded in the full matrix ring $\left(M_n(F), +, \cdot\right)$.

Proof
An embedment is an injective homomorphism. So it will be shown that there exists a one-to-one homomorphism between $(R_n(F), +, \circ)$ and $\left(M_n(F), +, \cdot\right)$. Now, we define the mapping

$\theta : \left(R_n(F), +, \circ\right) \to \left(M_n(F), +, \cdot\right)$ by

$$
\theta\left(\left\langle\begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\
 & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\
 & & & & a_{tt} & & & &
\end{array}\right\rangle\right) =
$$

$$
\begin{bmatrix}
a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\
0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\
a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\
0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\
a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt}
\end{bmatrix}
$$

That is, θ maps each rhotrix R_n in $\left(R_n(F),+,\circ\right)$ to its corresponding filled coupled matrix M_n in $\left(M_n(F),+,\cdot\right)$

Clearly, θ is a $1-1$, since $\theta\left(R_n\right)=\theta\left(S_n\right) \Rightarrow R_n = S_n$. Meaning that, no two different rhotrices in $\left(R_n(F),+,\circ\right)$ may have the same filled coupled matrices in $\left(M_n(F),+,\cdot\right)$. Furthermore, θ is a homomorphism, since for all $R_n, S_n \in \left(R_n(F),+,\circ\right)$, the images of θ in both equations $\theta\left(R_n+S_n\right)=\theta\left(R_n\right)+\theta\left(S_n\right)$ and $\theta\left(R_n\circ S_n\right)=\theta\left(R_n\right)\cdot\theta\left(S_n\right)$ are elements in $\left(M_n(F),+,\cdot\right)$. Thus θ is an injective homomorphism. Hence $(R_n(F),+,\circ)$ is embedded in $\left(M_n(F),+,\cdot\right)$.

Before the remarks below, it is interesting to recall from Garret (2008) that, a field is a commutative ring with unity element, on which each non-zero element possesses a multiplicative inverse element.

Remarks

(a) Let $R_n(K)$ be the collection of all rhotrices of size n with entries in an arbitrary field K. Let $+$ and $\circ$ denote respectively, the operations of rhotrix addition and row-column based multiplication. Then the triple $(R_n(K),+,\circ)$ is a non-commutative ring, but not a field. The main reason is that, some non-zero rhotrices in $(R_n(K),+,\circ)$ do not have multiplicative inverse rhotrix. Hence, $(R_n(K),+,\circ)$ is a non-commutative ring, but not a field, when K is a field.

(b) Let $GR_n(F)$ be the subset of the ring $(R_n(F),+,\circ)$ which consists of all invertible rhotrices of size n with entries from an arbitrary ring *F*. Then the pair $\left(GR_n(F),\circ\right)$ forms a group. This group $\left(GR_n(F),\circ\right)$ was studied in Mohammed and Okon (2016) as 'the *non-commutative general rhotrix group'*.

Some Properties of Non-Commutative Full Rhotrix Ring

(i) Let F be the ring of all integer or rational or real or complex or residue integer numbers. Let $(R_n(F),+,\circ)$ be the non-commutative rhotrix ring. An element $I_n \in (R_n(F),+,\circ)$ is called a unity element in $(R_n(F),+,\circ)$, if for all $X_n \in (R_n(F),+,\circ)$, we have $X_n \circ I_n = I_n \circ X_n = X_n$. Thus, the unity element in the non-commutative rhotrix ring $(R_n(F),+,\circ)$ is a rhotrix I_n given by equation (3).

(ii) Let F be the ring of all integer or rational or real or complex or residue integer numbers. Let $(R_n(F),+,\circ)$ be the non-commutative full rhotrix ring. An element $X_n \in (R_n(F),+,\circ)$ is called a unit element in $(R_n(F),+,\circ)$, if there exist an element $X_n^{-1} \in (R_n(F),+,\circ)$, such that $X_n \circ X_n^{-1} = X_n^{-1} \circ X_n = I_n$.

Thus, the units elements in the non-commutative full rhotrix ring $(R_n(F),+,\circ)$ are the invertible (or non-zero determinant) rhotrices. So, the pair $\left(GR_n(F),\circ\right)$ termed as '*non-commutative general rhotrix group'* in (Mohammed and Okon, 2016) forms a group of units in $(R_n(F),+,\circ)$.

Subrings of Non-Commutative Full Rhotrix Ring

Definition (Left triangular rhotrix)

Mohammed and Okon (2016) said that a rhotrix R_n is left triangular if all the elements in the right of the vertical diagonal are all zero.

Now, let us denote the set of all left triangular rhotrices of size n in $R_n(F)$ as $LTR_n(F)$. Thus,

$$LTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F \right\},$$

$$where \;\; a_{ij} = 0 \;\; if \;\; i < j \quad and \quad c_{lk} = 0 \; if \;\; l < k$$

.

Proposition (Mohammed and Okon, 2016)
If A_n and B_n are left triangular rhotrices, then their product $A_n \circ B_n$ is also a left triangular rhotrix.

Theorem
The triple $(LTR_n(F), +, \circ)$ which consist of the set of all left triangular rhotrices of size n over a ring F and together with the operations of rhotrix addition and row-column based multiplication is a subring of the non-commutative full rhotrix ring $(R_n(F), +, \circ)$.

Proof
It will be shown that the triple $(LTR_n(F), +, \circ)$ is closed under the operations of subtraction and multiplication as below.
Let

$$A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

and

$$B_n = \langle b_{ij}, d_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & 0 & & & \\ & & b_{31} & d_{21} & b_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ b_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & b_{t(t-2)} & d_{(t-1)(t-2)} & b_{(t-1)(t-1)} & 0 & 0 & & \\ & & & b_{t(t-1)} & d_{(t-1)(t-1)} & 0 & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle$$

be any two rhotrices of size n in $LTR_n(F)$. It follows that

$$A_n - B_n = \langle a_{ij} - b_{ij},\; c_{lk} - d_{lk} \rangle \in (LTR_n(F), +, \circ)$$

. Furthermore, following the above proposition in (Mohammed and Okon, 2016), the product $A_n \circ B_n \in (LTR_n(F), +, \circ)$. So the triple $(LTR_n(F), +, \circ)$ is closed under the operations of rhotrix subtraction and multiplication defined in the rhotrix ring $(R_n(F), +, \circ)$. Meaning that, the pair $(LTR_n(F), +)$ is a rhotrix subgroup of the rhotrix group $(R_n(F), +)$. Next, the pair $(LTR_n(F), \circ)$ is a rhotrix subsemigroup of the rhotrix semigroup $(R_n(F), \circ)$. Lastly, the triple $(LTR_n(F), +, \circ)$ satisfies the axiom that, the operation of rhotrix multiplication should be both left and right distributive over the operation of rhotrix addition, as in $(R_n(F), +, \circ)$. Hence, $(LTR_n(F), +, \circ)$ is a left triangular rhotrix subring of $(R_n(F), +, \circ)$.

Corollary
The left triangular rhotrix subring $(LTR_n(F), +, \circ)$ of the full rhotrix ring $(R_n(F), +, \circ)$ is embedded in the lower triangular matrix subring $(LTM_n(F), +, \cdot)$ of the full matrix ring $(M_n(F), +, \cdot)$.

Proof
It is required to show that there is a one-to-one homomorphism between $(LTR_n(F), +, \circ)$ and $(LTM_n(F), +, \cdot)$. So define a mapping $\varphi : (LTR_n(F), +, \circ) \to (LTM_n(F), +, \cdot)$ by

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$$\varphi\left(\left\langle \begin{array}{ccccccccccc} & & & & a_{11} & & & & & & \\ & & & a_{21} & c_{11} & 0 & & & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & & & \\ & & & & a_{tt} & & & & & & \end{array}\right\rangle\right) =$$

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots. & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & 0 \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & a_{t(t-1)} & 0 & a_{tt} \end{bmatrix}$$

That is φ maps each left triangular rhotrix R_n in $(LTR_n(F),+,\circ)$ to its corresponding filled coupled lower triangular matrix M_n in $(LTM_n(F),+,\cdot)$

Clearly, φ is a $1-1$, since $\forall R_n, S_n \in (LTR_n(F),+,\circ),\ \ \varphi(R_n)=\varphi(S_n) \Rightarrow R_n = S_n$.

Meaning that no two different rhotrices in $(LTR_n(F),+,\circ)$ may have the same filled coupled matrices in $(LTM_n(F),+,\cdot)$. Furthermore, φ is a homomorphism, since for all $R_n, S_n \in (LTR_n(F),+,\circ)$, the images of φ for both $\varphi(R_n+S_n)=\varphi(R_n)+\varphi(S_n)$ and $\varphi(R_n \circ S_n)=\varphi(R_n)\cdot\varphi(S_n)$ are elements in $(LTM_n(F),+,\cdot)$. Thus, φ is an injective homomorphism. Hence, $(LTR_n(F),+,\circ)$ is embedded in $(LTM_n(F),+,\cdot)$.

Definition (Right triangular rhotrix)

Mohammed and Okon (2016) said that a rhotrix R_n is right triangular if all the elements in the left of the vertical diagonal are all zero.

Now, let us denote the set of all right triangular rhotrices of size n in $R_n(F)$ as $RTR_n(F)$. Thus,

$$RTR_n(F) = \left\{ \begin{array}{l} \left\langle \begin{array}{cccccccccc} & & & & a_{11} & & & & & \\ & & & 0 & c_{11} & a_{12} & & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & & \\ & & & & a_{tt} & & & & & \end{array} \right\rangle : \\ a_{ij}, c_{lk} \in F \\ where \quad a_{ij}=0 \ \ if \ \ i>j \ \ and \ \ c_{lk}=0 \ if \ \ l>k \end{array} \right\},$$

Proposition (Mohammed and Okon, 2016)

If A_n and B_n are right triangular rhotrices, then their product $A_n \circ B_n$, is also a right triangular rhotrix.

Theorem

The triple $(RTR_n(F),+,\circ)$ which consist of the set of all right triangular rhotrices of size n over a ring F and together with the operations of rhotrix addition and row-column based multiplication is a subring of the non-commutative full rhotrix ring $(R_n(F),+,\circ)$.

Proof

It will be shown that the $(RTR_n(F),+,\circ)$ triple is closed under the operations of subtraction and multiplication as below:

Let

$$A_n = \langle a_{ij}, c_{lk} \rangle =$$

$$\left\langle \begin{array}{cccccccccc} & & & & a_{11} & & & & & \\ & & & 0 & c_{11} & a_{12} & & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & & \\ & & & & a_{tt} & & & & & \end{array} \right\rangle$$

and

$$B_n = \langle b_{ij}, d_{lk} \rangle =$$

$$\left\langle \begin{array}{c} b_{11} \\ 0 \quad d_{11} \quad b_{12} \\ 0 \quad 0 \quad b_{22} \quad d_{12} \quad b_{13} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad b_{1t} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad b_{(t-1)(t-1)} \quad d_{(t-2)(t-1)} \quad b_{(t-2)t} \\ 0 \quad d_{(t-1)(t-1)} \quad b_{(t-1)t} \\ b_{tt} \end{array} \right\rangle$$

be any two rhotrices of size n in $RTR_n(F)$. It follows that

$$A_n - B_n = \langle a_{ij} - b_{ij},\ c_{lk} - d_{lk} \rangle \in (RTR_n(F), +, \circ)$$

.

Furthermore, by the above proposition, we have $A_n \circ B_n \in (RTR_n(F), +, \circ)$.

So the triple $(RTR_n(F), +, \circ)$ is closed under the operations of rhotrix subtraction and multiplication.

Meaning that, the pair $(RTR_n(F), +)$ is a rhotrix subgroup of the rhotrix group $(R_n(F), +)$. Next, the pair $(RTR_n(F), \circ)$ is a rhotrix subsemigroup of the rhotrix semigroup $(R_n(F), \circ)$. Lastly, the triple $(RTR_n(F), +, \circ)$ satisfies the axiom that, the operation of rhotrix multiplication should be both left and right distributive over the operation of rhotrix addition, as in $(R_n(F), +, \circ)$.

Hence, $(RTR_n(F), +, \circ)$ is a right triangular rhotrix subring of $(R_n(F), +, \circ)$.

Corollary

The right triangular rhotrix subring $(RTR_n(F), +, \circ)$ of the full rhotrix ring $(R_n(F), +, \circ)$ is embedded in the upper triangular matrix subring $(UTM_n(F), +, \cdot)$ of the full matrix ring $(M_n(F), +, \cdot)$.

Proof

It is required to show that there is a one-to-one homomorphism between $(RTR_n(F), +, \circ)$ and $(UTM_n(F), +, \cdot)$.

So define a mapping $\varphi : (RTR_n(F), +, \circ) \to (UTM_n(F), +, \cdot)$ by

$$\varphi\left(\left\langle \begin{array}{c} a_{11} \\ 0 \quad c_{11} \quad a_{12} \\ 0 \quad 0 \quad a_{22} \quad c_{12} \quad a_{13} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad a_{1t} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad a_{(t-1)(t-1)} \quad c_{(t-2)(t-1)} \quad a_{(t-2)t} \\ 0 \quad c_{(t-1)(t-1)} \quad a_{(t-1)t} \\ a_{tt} \end{array} \right\rangle \right) =$$

$$\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & a_{1(t-1)} & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & 0 & c_{1(t-1)} & 0 \\ 0 & 0 & a_{22} & 0 & \cdots & \cdots & a_{2(t-1)} & 0 & a_{2t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdot & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & a_{(t-1)t} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a_{tt} \end{bmatrix}$$

That is φ maps each right triangular rhotrix R_n in $(RTR_n(F), +, \circ)$ to its corresponding filled coupled upper triangular matrix M_n in $(UTM_n(F), +, \cdot)$

Clearly, φ is a $1-1$, since $\varphi(R_n) = \varphi(S_n) \Rightarrow R_n = S_n,\ \forall R_n, S_n \in (RTR_n(F), +, \circ)$.

Meaning that no two different rhotrices in $(RTR_n(F), +, \circ)$ may have the same filled coupled matrices in $(UTM_n(F), +, \cdot)$. Furthermore, φ is a homomorphism, since for all $R_n, S_n \in (RTR_n(F), +, \circ)$, the images of φ for both $\varphi(R_n + S_n) = \varphi(R_n) + \varphi(S_n)$ and $\varphi(R_n \circ S_n) = \varphi(R_n) \cdot \varphi(S_n)$ are elements in $(UTM_n(F), +, \cdot)$. Thus, φ is an injective homomorphism. Hence, $(RTR_n(F), +, \circ)$ is embedded in $(UTM_n(F), +, \cdot)$.

Definition (Diagonal rhotrix)

Mohammed and Okon (2016) said that a rhotrix R_n is a diagonal rhotrix if all the elements in the vertical diagonal are non-zero, while others are zeros.

Now, let us denote the set of all diagonal rhotrices of size n in $R_n(F)$ as $DR_n(F)$. Thus,

$$DR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & 0 & & & \\ & & 0 & 0 & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F \right\}$$

Theorem

The triple $(DR_n(F), +, \circ)$ which consist of the set of all diagonal rhotrices of size n over a ring F and together with the operations of rhotrix addition and row-column based multiplication is a subring of the non-commutative full rhotrix ring $(R_n(F), +, \circ)$.

Proof

Let

$$A_n = \langle p_{ij}, q_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array} \right\rangle \in DR_n(F)$$

and

$$B_n = \langle r_{ij}, s_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & r_{11} & & & & \\ & & & 0 & s_{11} & 0 & & & \\ & & 0 & 0 & r_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & s_{(t-1)(t-1)} & 0 & & & \\ & & & & r_{tt} & & & & \end{array} \right\rangle \in DR_n(F)$$

It is simple to show that $DR_n(F)$ is closed under operations of rhotrix subtraction and multiplication, since

$$A_n - B_n = \left\langle \begin{array}{ccccccccc} & & & & p_{11} - r_{11} & & & & \\ & & & 0 & q_{11} - s_{11} & 0 & & & \\ & & 0 & 0 & p_{22} - r_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} - r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} - s_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} - r_{tt} & & & & \end{array} \right\rangle \in DR_n(F)$$

and

$$A_n \circ B_n = \left\langle \begin{array}{ccccccccc} & & & & p_{11} r_{11} & & & & \\ & & & 0 & q_{11} s_{11} & 0 & & & \\ & & 0 & 0 & p_{22} r_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} s_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} r_{tt} & & & & \end{array} \right\rangle \in DR_n(F)$$

So the triple $(DR_n(F), +, \circ)$ is closed under the operations of rhotrix subtraction and multiplication.

Meaning that, the pair $\left(DR_n(F),+\right)$ is a rhotrix subgroup of the rhotrix group $\left(R_n(F),+\right)$. Next, the pair $\left(DR_n(F),\circ\right)$ is a rhotrix subsemigroup of the rhotrix semigroup $\left(R_n(F),\circ\right)$. Lastly, the triple $\left(DR_n(F),+,\circ\right)$ satisfies the axiom that, the operation of rhotrix multiplication should be both left and right distributive over the operation of rhotrix addition, as in $\left(R_n(F),+,\circ\right)$.

Hence, $(DR_n(F),+,\circ)$ is a rhotrix subring of $(R_n(F),+,\circ)$

Corollary

The Diagonal rhotrix subring $(DR_n(F),+,\circ)$ of $(R_n(F),+,\circ)$ is embedded in the diagonal matrix subring $(DM_n(F),+,\cdot)$ of $(M_n(F),+,\cdot)$

Proof

Let $(DR_n(F),+,\circ)$ be a diagonal rhotrix subring of $(R_n(F),+,\circ)$ and let $(DM_n(F),+,\cdot)$ be a diagonal matrix subring of $(M_n(F),+,\cdot)$,

We define a mapping $\phi:(DR_n(F),+,\circ)\rightarrow(DM_n(F),+,\cdot)$ by

$$\phi\left(\left\langle\begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & 0 & c_{11} & 0 & & & & \\ & & & 0 & 0 & a_{22} & 0 & 0 & & & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\ 0 & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\ & & & 0 & 0 & 0 & 0 & 0 & & & \\ & & & & 0 & c_{(t-1)(t-1)} & 0 & & & & \\ & & & & & a_{tt} & & & & & \end{array}\right\rangle\right)=$$

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & a_{22} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{tt} \end{bmatrix}$$

Where ϕ mapped each diagonal rhotrix R_n in $DR_n(F)$, to its filled coupled diagonal matrix M_n in $DM_n(F)$.

Clearly, φ is a $1-1$, since $\forall R_n, S_n \in \left(DR_n(F),+,\circ\right),\ \varphi\left(R_n\right)=\varphi\left(S_n\right)\Rightarrow R_n=S_n$.

Meaning that no two different rhotrices in $\left(DR_n(F),+,\circ\right)$ will have the same filled coupled matrices in $\left(DM_n(F),+,\cdot\right)$. Furthermore, φ is a homomorphism, since for all $R_n, S_n \in \left(DR_n(F),+,\circ\right)$, the images of φ for both $\varphi\left(R_n+S_n\right)=\varphi\left(R_n\right)+\varphi\left(S_n\right)$ and $\varphi\left(R_n\circ S_n\right)=\varphi\left(R_n\right)\cdot\varphi\left(S_n\right)$ are elements in $\left(DM_n(F),+,\cdot\right)$. Thus, φ is an injective homomorphism.

Hence, $(DR_n(F),+,\circ)$ is embedded in $\left(DM_n(F),+,\cdot\right)$.

Definition (Scalar rhotrix)

Mohammed and Okon (2016) said that a rhotrix R_n is a scalar rhotrix if all the elements in the vertical diagonal are non-zero scalar, while others are zero(s). Scalar rhotrices are rhotrices of the form kI, where I is the identity rhotrix and k is a non-zero constant.

Now, let us denote the collection of all scalar rhotrices of size n in $R_n(F)$ as $KR_n(F)$.

Thus,

$$KR_n(F)=\left\{\left\langle\begin{array}{ccccccccccc} & & & & & k_{11} & & & & & \\ & & & & 0 & \kappa_{11} & 0 & & & & \\ & & & 0 & 0 & k_{22} & 0 & 0 & & & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\ 0 & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \\ & & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & & \\ & & & 0 & 0 & k_{(t-1)(t-1)} & 0 & 0 & & & \\ & & & & 0 & \kappa_{(t-1)(t-1)} & 0 & & & & \\ & & & & & k_{tt} & & & & & \end{array}\right\rangle : a_{ij}, c_{lk}\in F\right\}$$

Theorem

The triple $(KR_n(F),+,\circ)$ which consist of the set of all scalar rhotrices of size n over a ring F and together with the operations of rhotrix addition and row-column based multiplication is a subring of the non-commutative full rhotrix ring $(R_n(F),+,\circ)$.

The triple $(KR_n(F),+,\circ)$ is a rhotrix subring of $(R_n(F),+,\circ)$.

Proof

Let

$$A_n = \langle p_{ij}, p_{lk} \rangle = \left\langle \begin{array}{c} p_{11} \\ 0 \quad p_{11} \quad 0 \\ 0 \quad 0 \quad p_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad p_{11} \quad 0 \quad 0 \\ 0 \quad p_{11} \quad 0 \\ p_{11} \end{array} \right\rangle \in KR_n(F)$$

and

$$B_n = \langle r_{ij}, r_{lk} \rangle = \left\langle \begin{array}{c} r_{11} \\ 0 \quad r_{11} \quad 0 \\ 0 \quad 0 \quad r_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad r_{11} \quad 0 \quad 0 \\ 0 \quad r_{11} \quad 0 \\ r_{11} \end{array} \right\rangle \in KR_n(F)$$

It is simple follows that

$$A_n - B_n = \left\langle \begin{array}{c} p_{11}-r_{11} \\ 0 \quad p_{11}-r_{11} \quad 0 \\ 0 \quad 0 \quad p_{11}-r_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad p_{11}-r_{11} \quad 0 \quad 0 \\ 0 \quad p_{11}-r_{11} \quad 0 \\ p_{11}-r_{11} \end{array} \right\rangle$$

and

$$A_n \circ B_n = \left\langle \begin{array}{c} p_{11}r_{11} \\ 0 \quad p_{11}r_{11} \quad 0 \\ 0 \quad 0 \quad p_{11}r_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad p_{11}r_{11} \quad 0 \quad 0 \\ 0 \quad p_{11}r_{11} \quad 0 \\ p_{11}r_{11} \end{array} \right\rangle$$

are elements in $KR_n(F)$. So the triple $(KR_n(F),+,\circ)$ is closed under the operations of rhotrix subtraction and multiplication. Meaning that, the pair $(KR_n(F),+)$ is a rhotrix subgroup of the rhotrix group $(R_n(F),+)$. Next, the pair $(KR_n(F),\circ)$ is a rhotrix subsemigroup of the rhotrix semigroup $(R_n(F),\circ)$. Lastly, the triple $(KR_n(F),+,\circ)$ satisfies the axiom that, the operation of rhotrix multiplication should be both left and right distributive over the operation of rhotrix addition, as in $(R_n(F),+,\circ)$.

Hence, $(KR_n(F),+\circ)$ is a rhotrix subring of $(R_n(F),+,\circ)$.

Corollary

The scalar rhotrix subring $(KR_n(F),+,\circ)$ of the full rhotrix ring $(R_n(F),+,\circ)$ is embedded in the Scalar matrix subring $(KL_n(F),+,\cdot)$ of the full matrix ring $(M_n(F),+,\cdot)$.

Proof

Let $(KR_n(F),+,\circ)$ be a scalar rhotrix subring of $(R_n(F),+,\circ)$ and let $(KM_n(F),+,\cdot)$ be a scalar matrix subring of $(M_n(F),+,\cdot)$,

We define a mapping $\varphi : (KR_n(F),+,\circ) \to (KM_n(F),+,\cdot)$ by

$$\varphi\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & a_{11} & 0 & & & \\ & & 0 & 0 & a_{11} & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 0 & a_{11} & 0 & & & \\ & & & & a_{11} & & & & \end{array} \right\rangle\right) =$$

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & a_{11} & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & a_{11} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & a_{11} & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{11} \end{bmatrix}$$

Where φ maps each rhotrix R_n in $KR_n(F)$, to its filled coupled matrix M_n in $KL_n(F)$,

Clearly, φ is a $1-1$, since $\forall R_n, S_n \in (KR_n(F),+,\circ)$, $\varphi(R_n)=\varphi(S_n) \Rightarrow R_n = S_n$.

Meaning that no two different rhotrices in $(KR_n(F),+,\circ)$ will have the same filled coupled matrices in $(KM_n(F),+,\cdot)$. Furthermore, φ is a homomorphism, since for all $R_n, S_n \in (KR_n(F),+,\circ)$, the images of φ for both $\varphi(R_n+S_n)=\varphi(R_n)+\varphi(S_n)$ and $\varphi(R_n \circ S_n)=\varphi(R_n)\cdot\varphi(S_n)$ are elements in $(KM_n(F),+,\cdot)$. Thus, φ is an injective homomorphism. Hence, $(KR_n(F),+,\circ)$ is embedded in $(KM_n(F),+,\cdot)$.

Isomorphism between Some Subrings of Non-Commutative Full Rhotrix RING

Theorem

Let φ be a mapping from $(LTR_n(F),+,\circ)$ to $(RTR_n(F),+,\circ)$ defined by

$$\varphi\left(\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & \\ & & & a_{tt} & & & \end{array} \right\rangle\right) =$$

$$\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & 0 & c_{11} & a_{12} & & \\ & & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ & & \ldots & \ldots & \ldots & & \\ & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle$$

Then the mapping φ is a rhotrix ring isomorphism.

Proof

Let $(LTR_n(F),+,\circ)$ and $(RTR_n(F),+,\circ)$ be the ring of all left triangular rhotrices of size n and the ring of all right triangular rhotrices of size n respectively. By the hypothesis, there exists a mapping

$$\varphi:(LTR_n(F),+,\circ) \to (RTR_n(F),+,\circ) \ni \varphi(R_n) = \varphi(\langle a_{ij}, c_{lk}\rangle) = \langle a_{ji}, c_{kl}\rangle$$

This is a homomorphism, since

$$R_n = \langle a_{i_1 j_1}, c_{l_1 k_1}\rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & \\ & & & a_{tt} & & & \end{array} \right\rangle$$

and

$Q_n = \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle =$

$$\left\langle \begin{array}{ccccccc} & & & b_{11} & & & \\ & & b_{21} & d_{11} & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & b_{t(t-1)} & d_{(t-1)(t-1)} & 0 & & \\ & & & b_{tt} & & & \end{array} \right\rangle$$

implies that:

$$\varphi(R_n \circ Q_n) = \varphi\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle\right)$$

$$= \varphi\left(\sum_{i_2 j_1 = 1}^{t} a_{i_1 j_1} b_{i_2 j_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{l_1 k_1} d_{l_2 k_2} \right)$$

$$= \left(\sum_{i_2 j_1 = 1}^{t} a_{j_1 i_1} b_{j_2 i_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{k_1 l_1} d_{k_2 l_2} \right)$$

$$= \left\langle a_{j_1 i_1}, c_{k_1 l_1} \right\rangle \circ \left\langle b_{j_2 i_2}, d_{k_2 l_2} \right\rangle$$

$$= \varphi\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle\right) \circ \varphi\left(\left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle\right)$$

$$= \varphi(R_n) \circ \varphi(Q_n)$$

Also,

$$\varphi(R_n + Q_n) = \varphi\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle + \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle\right) =$$

$$\varphi\left(\left\langle a_{i_1 j_1} + b_{i_2 j_2}, c_{l_1 k_1} + d_{l_2 k_2} \right\rangle\right) = \left\langle a_{j_1 i_1} + b_{j_2 i_2}, c_{k_1 l_1} + d_{k_2 l_2} \right\rangle$$

$$= \left\langle a_{j_1 i_1}, c_{k_1 l_1} \right\rangle + \left\langle b_{j_2 i_2}, d_{k_2 l_2} \right\rangle$$

$$= \varphi\left(\left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle\right) + \varphi\left(\left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle\right)$$

$$= \varphi(R_n) + \varphi(Q_n)$$

Finally, φ is a bijection because the kernel and image of φ are respectively as follows:

$$\ker(\varphi) = \left\{ I_n \in (LTR_n(F), +, \circ) : \varphi(I_n) = I_n^T \in (RTR_n(F), +, \circ) \right\}$$

and

$$\mathrm{Im}(\varphi) = \left\{ R_n \in (RTR_n(F), +, \circ) : \varphi(S_n) = R_n, \forall S_n \in (LTR_n(F), +, \circ) \right\} = (RTR_n(F), +, \circ).$$

Hence, $(LTR_n(F), +, \circ) \cong (RTR_n(F), +, \circ).$

Conclusion

The concept of non-commutative rings of rhotrices and their generalization as non-commutative full rhotrix ring had been introduced. The subrings of the non-commutative full rhotrix ring were determined. It was also shown that the non-commutative full rhotrix ring is embedded on the well-known non-commutative full matrix ring. Furthermore, it was also established that a particular subring of the non-commutative full rhotrix ring is embedded on a particular subring of the non-commutative full matrix ring. At the end, we investigated some isomorphic relationship between some subrings of the non-commutative full rhotrix ring. In the future, it may be interesting to consider a number of topics on non-commutative rhotrix ring such as computing finite non-commutative rings of rhotrices, investigation of its ideals and quotients rings, and also mappings of non-commutative rings of rhotrices.

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