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RHOTRIX NORMAL SUBGROUPS AND QUOTIENT GROUPS

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ABSTRACT
This paper uncovers the normal subgroups of the non-commutative general rhotrix group and establishes their corresponding quotient groups. The ideas are presented to serve as an extension to the recent work by Mohammed and Okon on subgroups of the non-commutative general rhotrix group. In the process, a number of theorems are developed and concrete example shown.

Keywords: Rhotrix, Group, Subgroup, Normal subgroup, Quotient group

INTRODUCTION

Since the concept of rhotrix was initiated by Ajibade (2003) as an extension of ideas on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon (1998), there have been many expression of interest by researchers in the usage of rhotrix set as an underlying set in the study of various forms of algebraic structures (see Aminu *et al.* (2017), Mohammed (2007a and 2007b), Mohammed and Sani (2011), Mohammed *et al.*(2014), Mohammed and Okon (2016) and Mohammed and Balarabe (2017)). The initial algebra and analysis of rhotrices of size 3 were discussed in (Ajibade, 2003). Following this, Sani (2004) defined a rhotrix R of size n as a rhomboidal array of numbers which can be expressed as a couple of two square matrices A and C of sizes $(t\times t)$ and $(t-1)\times(t-1)$, where $t=\frac{n+1}{2}$ and $n\in 2Z^{+}+1$. That is,

$$R_n=\left\langle A_{t\times t},C_{(t-1)\times(t-1)}\right\rangle=\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle$$

$$=\left\langle\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(t-1)} & a_{1t}\\ a_{21} & a_{22} & \cdots & a_{2(t-1)} & a_{2t}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{(t-1)1} & a_{(t-1)2} & \cdots & a_{(t-1)(t-1)} & a_{(t-1)t}\\ a_{t1} & a_{t2} & \cdots & a_{t(t-1)} & a_{tt}\end{bmatrix},\begin{bmatrix} c_{11} & \cdots & c_{1(t-1)}\\ \cdots & \cdots & \cdots\\ c_{(t-1)1} & \cdots & c_{(t-1)(t-1)}\end{bmatrix}\right\rangle,$$

where $[a_{ij}]$ and $[c_{lk}]$ are called the major and minor matrices of R_n respectively. The set $R_n(F)$, consisting of all such collections of rhotrices with entries from an arbitrary field F is given as:

$$R_n(F)=\left\{\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle : a_{ij}\in F, c_{lk}\in F\right\}$$

where $1\le i,j\le t$,

$$1\le l,k\le t-1;\ t=\frac{n+1}{2}\ and\ n\in 2Z^{+}+1.$$

A row-column method for multiplication of two rhotrices R_n, Q_n having the same size was defined in (Sani, 2007) as:

$$R_n\circ Q_n=\left\langle a_{i_1j_1},c_{l_1k_1}\right\rangle\circ\left\langle b_{i_2j_2},d_{l_2k_2}\right\rangle$$

$$=\left\langle\sum_{i_2j_1}^{t}(a_{i_1j_1}b_{i_2j_2}),\sum_{l_2k_1}^{t-1}(c_{l_1k_1}d_{l_2k_2})\right\rangle.$$

It was noted that this row-column method for rhotrix multiplication is non-commutative, but associative. The identity rhotrix for any real rhotrix of size n was given as:

$$I_n=\left\langle I_{t\times t},I_{(t-1)\times(t-1)}\right\rangle=\left\langle\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle.$$

The determinant of a rhotrix $R$of size n was also defined as $\det(R_n) = \det\left\langle a_{ij}, c_{lk} \right\rangle = \det(A_{t\times t}) \cdot \det (C_{(t-1)\times(t-1)})$; and that R_n is invertible if and only if $\det(R_n) \neq 0.$ Furthermore, for any rhotrix $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$, the transpose of R_n was defined as $R_n^T = \left\langle a_{ji}, c_{kl} \right\rangle$. It was also shown in (Sani, 2007) that $\det(R_n \circ Q_n) = \det(R_n) \circ \det(Q_n) = \det(R_n) \cdot \det(Q_n)$ and $(R_n \circ Q_n)^T = (Q_n)^T \circ (R_n)^T.$

Throughout this paper, let $GR_n(F)$ denote the set of all invertible rhotrices of size n with entries from a field F That is,

$$GR_n(F) = \left\{ \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots\cdot & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle : \begin{array}{l} \\ a_{ij}, c_{lk} \in F \text{ and } \det([a_{ij}]) \neq 0 \neq \det([c_{lk}]) \end{array} \right\}$$

where $1 \le i, j \le t,$

$$1 \le l, k \le t-1;\ t = \frac{n+1}{2} \text{ and } n \in 2Z^+ + 1.$$

The pair $(GR_n(F), \circ)$ was shown in (Mohammed and Okon, 2016) as the non-commutative general rhotrix group of size *n* over an arbitrary field *F*. The subgroups of this non-commutative general rhotrix group of size n over an arbitrary field F were identified in their work. These include: the special rhotrix group; upper and lower triangular rhotrix groups; diagonal and scalar rhotrix group.Now, the interest is to uncover the normal subgroups and quotient groups of non-commutative general rhotrix group. To the best of our knowledge, the normal subgroupsof non-commutative general rhotrix group and their corresponding quotient groupshad never been considered in the literature of rhotrix theory as a whole.

A subgroup H of a group G is called normal if $xH = Hx,\ \forall x \in G$. Furthermore, if H is a normal subgroup of a group G, then there exist a group $G/H,$ whose elements are the distinct left (right) cosets of H in G (Fraleigh, 2003). In line with this idea, it was shown in (Seymour, 2005) that, the Special Linear group $SL_n(F)$ consisting of all invertible $n \times n$ dimensional matrices with determinant as 1, over an arbitrary field $F$is a normal subgroup of the General Linear group $GL_n(F)$ of degree n over an arbitrary field F. In this paper, an analogous of this result will beestablished in the context of rhotrix theory. That is, it will be shown that, the set $SR_n(F)$, consisting of all invertible rhotrices of size $n$with determinant as 1, over an arbitrary field F,is a normal subgroup ofthe Non-commutative General Rhotrix Group of size n over an arbitrary field F $GR_n(F).$ Furthermore, an analogous result of (Fraleigh, 2003)will also be uncovered for non-commutative rhotrix group. This means, it would be shown that, there exist a quotient group $GR_n(F)/SR_n(F)$ whose elements are the distinct left (right) cosets of $SR_n(F)$ in $GR_n(F).$ Moreover, in order to minimize the abstractions, construction of concrete examples of normal subgroup of a given non-commutative rhotrix group andits corresponding quotient group are given, along with their group tables.

This work is significant, because it introduces the notions of normal subgroups and quotient groups having rhotrix set as underlying set. Part aside, the concrete examples given in thiswork, can furtherserve the purpose of reducing the abstractionsduring teaching and learning of these concepts in abstract algebra

The Non-Commutative General Rhotrix Group and Its Special Rhotrix Subgroup

The theorem 1 and theorem 2 below are recorded from (Mohammed and Okon, 2016) and will be of help in our discussions in subsequent sections

Theorem 1

Let $GR_n(F)$ be the set of all invertible rhotrices with entries from an arbitrary field *F* and let $\circ$ be the row-column method for rhotrix multiplication. Then, the pair $(GR_n(F), \circ)$ is a non-commutative general rhotrix group of size n over *F*.

Proof

We shall show that the pair $(GR_n(F), \circ)$ is a group under the binary operation of row-column multiplication of rhotrices. i.e. we shall show that the following group axioms are satisfied:

(i) Closure: for any two rhotrices of A_n , $B_n \in GR_n(F)$,

$\det(A_n) \neq 0 \Rightarrow A_n$ is invertible, and

$\det(B_n) \neq 0 \Rightarrow B_n$ is invertible. Now,

$A_n \circ B_n \in GR_n(F)$ since

$\det(A_n \circ B_n) = \det(A_n) \cdot \det(B_n) \neq 0$

Thus, $GR_n(F)$ is closed under the group binary operation.

(ii) Associativity:

$\forall A_n, B_n \text{ and } C_n \in GR_n(F)$

$(A_n \circ B_n) \circ C_n = A_n \circ (B_n \circ C_n)$

as shown in (Sani, 2004) that row-column multiplication of rhotrices is non-commutative but associative.

(iii) Existence of identity: for each $A_n \in GR_n(F), \exists$

$$I_n = \langle I_{t\times t}, I_{(t-1)\times(t-1)} \rangle =$$

$$\left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & & \cdots & \cdots & 1 & \cdots & \cdots & & 0 \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & 0 & 1 & 0 & & & \\ & & & & 1 & & & & \end{array} \right\rangle \in GR_n(F)$$

such that

$$I_n \circ R_n = R_n \circ I_n = R_n, \ \forall R_n \in GR_n(F)$$

(iv) Existence of inverse: for each $A_n \in GR_n(F), \exists$ $A_n^{-1} \in GR_n(F)$ such that $A_n \circ A_n^{-1} = I_n \in GR_n(F)$. So $A_n^{-1} \in GR_n(F)$.

Hence, $(GR_n(F), \circ)$ is a group under the operation of row-column method for multiplication of rhotrices.

Theorem 2

Let $SR_n(F)$ be the subset of $GR_n(F)$ consisting of all rhotrices of size n having determinant as 1and let $\circ$ be the non-commutative method for rhotrix multiplication. Then the pair $(SR_n(F), \circ)$ is a special rhotrix subgroup of $(GR_n(F), \circ)$.

Proof

Since $I_n \in SR_n(F)$, then $SR_n(F) \neq \emptyset$.

Now, Let A_n and $B_n \in SR_n(F)$,

Then it follows that, det(A_n) =1 $\neq 0$ and det(B_n) =1 $\neq 0$ respectively. This implies that for each A_n and $B_n \in SR_n(F), \ \exists \ A_n^{-1}$ and $B_n^{-1} \in SR_n(F) \ni A_n \circ B_n^{-1} \in SR_n(F)$ and

$$\det(A_n \circ B_n^{-1}) = \det(A_n) \circ \det(B_n^{-1}) = 1 \circ 1^{-1}$$

Hence $(SR_n(F), \circ)$ is a subgroup of $(GR_n(F), \circ)$.

The Special Normal Subgroup and Quotient Group

Theorem 3

Let $(GR_n(F), \circ)$ be the group of all invertible rhotrices of size n with entries from an arbitrary field F under row-column method of rhotrix multiplication′ ∘ ′. Let $SR_n(F)$ be the subset of $(GR_n(F), \circ)$ consisting of all rhotrices having determinant as 1. Then the pair $(SR_n(F), \circ)$ is a special normal subgroup of $(GR_n(F), \circ)$.

Proof

From theorem 2 above, the pair $(SR_n(F), \circ)$ is a subgroup of $(GR_n(F), \circ)$. Now, it remains to show that $(SR_n(F), \circ)$ is a normal subgroup of $(GR_n(F), \circ)$.

For any rhotrix $X_n \in GR_n(F)$ and any rhotrix $A_n \in SR_n(F), \ \det(X_n^{-1} \circ A_n \circ X_n) = 1.$

Hence, $X_n^{-1} \circ A_n \circ X_n$ belongs to $SR_n(F)$.

So, $SR_n(F)$ is a special normal subgroup of $GR_n(F)$.

Theorem 4

Let $(GR_n(F), \circ)$ be the non-commutative general rhotrix group and let $(SR_n(F), \circ)$ be the special normal subgroup of $(GR_n(F), \circ)$. Then the cosets of $SR_n(F)$ in $GR_n(F)$ forms a special quotient group $GR_n(F) / SR_n(F)$ under rhotrix coset multiplication, as defined by

$$\left(A_n \circ SR_n(F)\right) \circ \left(B_n \circ SR_n(F)\right) = (A_n \circ B_n) \circ SR_n(F).$$

Proof

The rhotrix coset multiplication is well –defined, since

$$\left(A_n \circ SR_n(F)\right) \circ \left(B_n \circ SR_n(F)\right)$$
$$= A_n \circ \left(SR_n(F) \circ B_n\right) \circ SR_n(F)$$
$$= A_n \circ \left(B_n \circ SR_n(F)\right) \circ SR_n(F)$$
$$= (A_n \circ B_n) \circ \left(SR_n(F) \circ SR_n(F)\right) = (A_n \circ B_n) \circ SR_n(F)$$

using the fact that $SR_n(F)$ is a special normal subgroup of $GR_n(F)$, so

$SR_n(F) \circ B_n = B_n \circ SR_n(F)$ and

$\left(SR_n(F) \circ SR_n(F)\right) = SR_n(F)$.

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $GR_n(F)$. Notice that $SR_n(F)$ is the identity element of $GR_n(F) / SR_n(F)$, since

$$\left(A_n \circ SR_n(F)\right) \circ SR_n(F) = A_n \circ \left(SR_n(F) \circ SR_n(F)\right)$$
$$= A_n \circ SR_n(F)$$

and

$SR_n(F)\circ\left(A_n\circ SR_n(F)\right)$
$=\left(SR_n(F)\circ A_n\right)\circ SR_n(F)$
$=\left(A_n\circ SR_n(F)\right)\circ SR_n(F)=A_n\circ SR_n(F)$

Lastly, $A_n^{-1}\circ SR_n(F)$ is the inverse of $A_n\circ SR_n(F),$ since

$\left(A_n^{-1}\circ SR_n(F)\right)\circ\left(A_n\circ SR_n(F)\right)$
$=\left(A_n^{-1}\circ A_n\right)\circ SR_n(F)$
$=I_n\circ SR_n(F)=SR_n(F)$

and

$\left(A_n\circ SR_n(F)\right)\circ\left(A_n^{-1}\circ SR_n(F)\right)$
$=\left(A_n\circ A_n^{-1}\right)\circ SR_n(F)$
$=I_n\circ SR_n(F)=SR_n(F)$

Hence, $GR_n(F)/SR_n(F)$ is a quotient group under operation of multiplication of cosets of $SR_n(F)$ in $GR_n(F).$

OTHER RESULTS

Theorem 5

Let $(SR_n(F),\circ)$ be the non-commutative special rhotrix group and let $SRTR_n(F)$ be a subset of $SR_n(F),$ consisting of all right triangular rhotrices with determinant as 1, then $SRTR_n(F)$ is a normal subgroup of $SR_n(F)$ and there exist the quotient group $SR_n(F)/SLTR_n(F)$ under operation of coset multiplication, defined by

$\left(A_n\circ SRTR_n(F)\right)\circ\left(B_n\circ SRTR_n(F)\right)=$
$(A_n\circ B_n)\circ SRTR_n(F).$

Proof

First, it will be shown that $SRTR_n(F)$ is a normal subgroup of $SR_n(F)$ and finally, show that the cosets of $SRTR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets.

Let

$SRTR_n(F)=$

$$\left\{\left\langle\begin{matrix} &&&&&a_{11}&&&\\ &&&&0_{21}&c_{11}&a_{12}&&\\ &&&0_{31}&0_{21}&a_{22}&c_{12}&a_{13}&\\ &\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&&\\ 0_{t1}&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&a_{1t}\\ &\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&&\\ &&0_{t(t-2)}&0_{(t-1)(t-2)}&a_{(t-1)(t-1)}&c_{(t-2)(t-1)}&a_{(t-2)t}&&\\ &&&0_{t(t-1)}&c_{(t-1)(t-1)}&a_{(t-1)t}&&&\\ &&&&a_{tt}&&&& \end{matrix}\right\rangle : a_{ij},c_{lk}\in F,\det([a_{ij}])=1=\det([c_{lk}])\right\},$$

where $1\le i,\ j\le t,$

$1\le l,k\le t-1;\ t=\frac{n+1}{2}\ ,\ n\in 2Z^{+}+1.$

Since $\det(I_n)=1,\ I_n$ belongs to $SRTR_n(F).$ If A_n and B_n belong to $SRTR_n(F),$ then,

$\det(A_n\circ B_n)=\det(A_n)\circ\det(B_n)=(1)\circ(1)=1,$
$\Rightarrow A_n\circ B_n\in SRTR_n(F).$

Also, $\det(A_n^{-1})=\dfrac{1}{\det(A_n)}=\dfrac{1}{1}=1.$

$\Rightarrow A_n^{-1}\in SRTR_n(F).$ Associativity law holds in $SRTR_n(F)$, since by virtue of associativity in $SR_n(F).$ Hence, $SRTR_n(F)$ is a subgroup of $SR_n(F).$

Now, for any special rhotrix $X_n\in SR_n(F)$ and any special right triangular rhotrix $A_n\in SRTR_n(F),$ we have $\det(X_n^{-1}\circ A_n\circ X_n)=1.$ Hence, $X_n^{-1}\circ A_n\circ X_n$ belongs to $SRTR_n(F).$ So, $SRTR_n(F)$ is a special right triangular normal subgroup of $SR_n(F).$

Finally, to show that the cosets of $SRTR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets is as follows: Let the operation of coset multiplication in $SRTR_n(F)$ be defined by

$\left(A_n\circ SRTR_n(F)\right)\circ\left(B_n\circ SRTR_n(F)\right)=$
$(A_n\circ B_n)\circ SRTR_n(F)$

Then the rhotrix coset multiplication is well –defined, since

$\left(A_n\circ SRTR_n(F)\right)\circ\left(B_n\circ SRTR_n(F)\right)=$
$A_n\circ\left(SRTR_n(F)\circ B_n\right)\circ SRTR_n(F)$

$= A_n \circ \left(B_n \circ SRTR_n(F)\right) \circ SRTR_n(F)$
$= (A_n \circ B_n) \circ \left(SRTR_n(F) \circ SRTR_n(F)\right)$
$= (A_n \circ B_n) \circ SRTR_n(F)$

using the fact that $SRTR_n(F)$ is a special right triangular normal subgroup of $SR_n(F)$, so

$SRTR_n(F) \circ B_n = B_n \circ SRTR_n(F)$ and

$\left(SRTR_n(F) \circ SRTR_n(F)\right) = SRTR_n(F)$.

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $SR_n(F)$. Notice that $SRTR_n(F)$ is the identity element of $SR_n(F)/SRTR_n(F)$, since

$\left(A_n \circ SRTR_n(F)\right) \circ SRTR_n(F)$
$= A_n \circ \left(SRTR_n(F) \circ SRTR_n(F)\right) = A_n \circ SRTR_n(F)$

and

$SRTR_n(F) \circ \left(A_n \circ SRTR_n(F)\right)$
$= \left(SRTR_n(F) \circ A_n\right) \circ SRTR_n(F)$
$= \left(A_n \circ SRTR_n(F)\right) \circ SRTR_n(F) = A_n \circ SRTR_n(F)$

Lastly, $A_n^{-1} \circ SRTR_n(F)$ is the inverse of $A_n \circ SRTR_n(F)$, since

$\left(A_n^{-1} \circ SRTR_n(F)\right) \circ \left(A_n \circ SRTR_n(F)\right)$
$= \left(A_n^{-1} \circ A_n\right) \circ SRTR_n(F) = I_n \circ SRTR_n(F)$
$= SRTR_n(F)$

and

$\left(A_n \circ SRTR_n(F)\right) \circ \left(A_n^{-1} \circ SRTR_n(F)\right)$
$= \left(A_n \circ A_n^{-1}\right) \circ SRTR_n(F) = I_n \circ SRTR_n(F)$
$= SRTR_n(F)$

Hence, $SR_n(F)/SRTR_n(F)$ is a quotient group under operation of coset multiplication.

Theorem 6

Let $(SR_n(F), \circ)$ be the non-commutative special rhotrix group and let $SLTR_n(F)$ be a subset of $SR_n(F)$, consisting of all left triangular rhotrices with determinant as 1, then $SLTR_n(F)$ is a normal subgroup of $SR_n(F)$ and there exist the quotient group $SR_n(F)/SLTR_n(F)$ under operation of coset multiplication, defined by

$\left(A_n \circ SLTR_n(F)\right) \circ \left(B_n \circ SLTR_n(F)\right)$
$= (A_n \circ B_n) \circ SLTR_n(F).$

Proof

First, it will be shown that $SLTR_n(F)$ is a normal subgroup of $SR_n(F)$ and finally, show that the cosets of $SLTR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets.

Let

$$SLTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & 0_{12} & 0_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0_{(t-2)(t-1)} & 0_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det([a_{ij}]) = 1 = \det([c_{lk}]) \right\},$$

where $1 \le i, j \le t,$

$1 \le l, k \le t-1;\ t = \dfrac{n+1}{2}$, $n \in 2Z^+ + 1.$

Since $\det(I_n) = 1$, I_n belongs to $SLTR_n(F)$. If A_n and B_n belong to $SLTR_n(F)$, then,

$\det(A_n \circ B_n) = \det(A_n) \circ \det(B_n) = (1) \circ (1) = 1,$
$\Rightarrow A_n \circ B_n \in SLTR_n(F).$

Also, $\det(A_n^{-1}) = \dfrac{1}{\det(A_n)} = \dfrac{1}{1} = 1.$

$\Rightarrow A_n^{-1} \in SLTR_n(F)$. $SLTR_n(F)$ is associative, since associativity law holds in $SR_n(F)$. Thus, $SLTR_n(F)$ is a subgroup of $SR_n(F)$.

Now, for any special rhotrix $X_n \in SR_n(F)$ and any special left triangular rhotrix $A_n \in SLTR_n(F)$, we have $\det(X_n^{-1} \circ A_n \circ X_n) = 1$. Hence, $X_n^{-1} \circ A_n \circ X_n$ belongs to $SLTR_n(F)$. So, $SLTR_n(F)$ is a special left triangular normal subgroup of $SR_n(F)$.

Finally, to show that the cosets of $SLTR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets is as follows: Let the operation of coset multiplication in $SLTR_n(F)$ be defined by

$$\left(A_n \circ SLTR_n(F)\right) \circ \left(B_n \circ SLTR_n(F)\right) = (A_n \circ B_n) \circ SLTR_n(F)$$

Then the rhotrix coset multiplication is well –defined, since

$$\left(A_n \circ SLTR_n(F)\right) \circ \left(B_n \circ SLTR_n(F)\right)$$
$$= A_n \circ \left(SLTR_n(F) \circ B_n\right) \circ SLTR_n(F)$$
$$= A_n \circ \left(B_n \circ SLTR_n(F)\right) \circ SLTR_n(F)$$
$$= (A_n \circ B_n) \circ \left(SLTR_n(F) \circ SLTR_n(F)\right)$$
$$= (A_n \circ B_n) \circ SLTR_n(F)$$

using the fact that $SLTR_n(F)$ is a special left triangular normal subgroup of $SR_n(F)$, so

$SLTR_n(F) \circ B_n = B_n \circ SLTR_n(F)$ and $\left(SLTR_n(F) \circ SLTR_n(F)\right) = SLTR_n(F)$.

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $SR_n(F)$. Notice that $SLTR_n(F)$ is the identity element of $SR_n(F) \big/ SLTR_n(F)$, since

$$\left(A_n \circ SLTR_n(F)\right) \circ SLTR_n(F)$$
$$= A_n \circ \left(SLTR_n(F) \circ SLTR_n(F)\right) = A_n \circ SLTR_n(F)$$

and

$$SLTR_n(F) \circ \left(A_n \circ SLTR_n(F)\right)$$
$$= \left(SLTR_n(F) \circ A_n\right) \circ SLTR_n(F)$$
$$= \left(A_n \circ SLTR_n(F)\right) \circ SLTR_n(F) = A_n \circ SLTR_n(F)$$

Lastly, $A_n^{-1} \circ SLTR_n(F)$ is the inverse of $A_n \circ SLTR_n(F)$, since

$$\left(A_n^{-1} \circ SLTR_n(F)\right) \circ \left(A_n \circ SLTR_n(F)\right)$$
$$= \left(A_n^{-1} \circ A_n\right) \circ SLTR_n(F) = I_n \circ SLTR_n(F)$$
$$= SLTR_n(F)$$

and

$$\left(A_n \circ SLTR_n(F)\right) \circ \left(A_n^{-1} \circ SLTR_n(F)\right)$$
$$= \left(A_n \circ A_n^{-1}\right) \circ SLTR_n(F) = I_n \circ SLTR_n(F)$$
$$= SLTR_n(F)$$

Hence, $SR_n(F) \big/ SLTR_n(F)$ is a quotient group under operation of coset multiplication.

Theorem 7

Let $(SR_n(F), \circ)$ be the non-commutative special rhotrix group and let $SDR_n(F)$ be a subset of $SR_n(F)$, consisting of all diagonal rhotrices with determinant as 1, then $SDR_n(F)$ is a normal subgroup of $SR_n(F)$ and there exist the quotient group $SR_n(F)/SDR_n(F)$ under operation of coset multiplication, defined by

$$\left(A_n \circ SDR_n(F)\right) \circ \left(B_n \circ SDR_n(F)\right) = (A_n \circ B_n) \circ SDR_n(F).$$

Proof

First, it will be shown that $SDR_n(F)$ is a normal subgroup of $SR_n(F)$ and finally, show that the cosets of $SDR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets. Let

$$SDR_n(F) = \left\{ \left\langle \begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & 0_{21} & c_{11} & 0_{12} & & & & \\ & & & 0_{31} & 0_{21} & a_{22} & 0_{12} & 0_{13} & & & \\ & & \dots & \dots & \dots & \dots & \dots & \dots & & & \\ 0_{t1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0_{1t} \\ & & \dots & \dots & \dots & \dots & \dots & \dots & & & \\ & & & 0_{t(t-2)} & 0_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0_{(t-2)(t-1)} & 0_{(t-2)t} & & & \\ & & & & 0_{t(t-1)} & c_{(t-1)(t-1)} & 0_{(t-1)t} & & & & \\ & & & & & a_{tt} & & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det([a_{ij}]) = 1 = \det([c_{lk}]) \right\},$$

where $1 \le i,\ j \le t$,

$$1 \le l, k \le t-1;\ t = \frac{n+1}{2},\ n \in 2Z^+ + 1.$$

Since $\det(I_n) = 1$, I_n belongs to $SDR_n(F)$. If A_n and B_n belong to $SDR_n(F)$, then,

$$\det(A_n \circ B_n) = \det(A_n) \circ \det(B_n) = (1) \circ (1) = 1,$$

$\Rightarrow A_n \circ B_n \in SDR_n(F).$ Associativity law holds in $SDR_n(F)$ as in $SR_n(F).$ Also, $\det(A_n^{-1}) = \dfrac{1}{\det(A_n)} = \dfrac{1}{1} = 1.$

$\Rightarrow A_n^{-1} \in SDR_n(F).$ Thus, $SDR_n(F)$ is a subgroup of $SR_n(F).$

Now, for any special rhotrix $X_n \in SR_n(F)$ and any special diagonal rhotrix $A_n \in SDR_n(F),$ we have $\det(X_n^{-1} \circ A_n \circ X_n) = 1.$ Hence, $X_n^{-1} \circ A_n \circ X_n$ belongs to $SDR_n(F).$ So, $SDR_n(F)$ is a special diagonal normal subgroup of $SR_n(F).$

Finally, to show that the cosets of $SDR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets is as follows: Let the operation of coset multiplication in $SDR_n(F)$ be defined by

$$\left(A_n \circ SDR_n(F)\right) \circ \left(B_n \circ SDR_n(F)\right)$$
$$= (A_n \circ B_n) \circ SDR_n(F)$$

Then the rhotrix coset multiplication is well –defined, since

$$\left(A_n \circ SDR_n(F)\right) \circ \left(B_n \circ SDR_n(F)\right)$$
$$= A_n \circ \left(SDR_n(F) \circ B_n\right) \circ SDR_n(F)$$
$$= A_n \circ \left(B_n \circ SDR_n(F)\right) \circ SDR_n(F)$$
$$= (A_n \circ B_n) \circ \left(SDR_n(F) \circ SDR_n(F)\right)$$
$$= (A_n \circ B_n) \circ SDR_n(F)$$

using the fact that $SDR_n(F)$ is a special diagonal normal subgroup of $SR_n(F),$ so

$SDR_n(F) \circ B_n = B_n \circ SDR_n(F)$ and

$\left(SDR_n(F) \circ SDR_n(F)\right) = SDR_n(F)$.

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $SR_n(F).$ Notice that $SDR_n(F)$ is the identity element of $SR_n(F) / SDR_n(F)$, since

$$\left(A_n \circ SDR_n(F)\right) \circ SDR_n(F) =$$
$$A_n \circ \left(SDR_n(F) \circ SDR_n(F)\right) = A_n \circ SDR_n(F)$$

and

$$SDR_n(F) \circ \left(A_n \circ SDR_n(F)\right)$$
$$= \left(SDR_n(F) \circ A_n\right) \circ SDR_n(F)$$
$$= \left(A_n \circ SDR_n(F)\right) \circ SDR_n(F) = A_n \circ SDR_n(F)$$

Lastly, $A_n^{-1} \circ SDR_n(F)$ is the inverse of $A_n \circ SDR_n(F),$ since

$$\left(A_n^{-1} \circ SDR_n(F)\right) \circ \left(A_n \circ SDR_n(F)\right)$$
$$= \left(A_n^{-1} \circ A_n\right) \circ SDR_n(F) = I_n \circ SDR_n(F) = SDR_n(F)$$

and

$$\left(A_n \circ SDR_n(F)\right) \circ \left(A_n^{-1} \circ SDR_n(F)\right)$$
$$= \left(A_n \circ A_n^{-1}\right) \circ SDR_n(F) = I_n \circ SDR_n(F) = SDR_n(F)$$

Hence, $SR_n(F) / SDR_n(F)$ is a quotient group under operation of coset multiplication.

Theorem 8

Let $(SR_n(F), \circ)$ be the non-commutative special rhotrix group and let $SKR_n(F)$ be a subset of $SR_n(F),$ consisting of all scalar rhotrices with determinant as 1, then $SKR_n(F)$ is a normal subgroup of $SR_n(F)$ and there exist the quotient group $SR_n(F)/SKR_n(F)$ under operation of coset multiplication, defined by

$$\left(A_n \circ SKR_n(F)\right) \circ \left(B_n \circ SKR_n(F)\right)$$
$$= (A_n \circ B_n) \circ SKR_n(F).$$

Proof

First, it will be shown that $SKR_n(F)$ is a normal subgroup of $SR_n(F)$ and finally, show that the cosets of $SKR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets.

Let

$$SKR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & k_{11} & & & & \\ & & & 0_{21} & \kappa_{11} & 0_{12} & & & \\ & & 0_{31} & 0_{21} & k_{22} & 0_{12} & 0_{13} & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0_{t1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ & & 0_{t(t-2)} & 0_{(t-1)(t-2)} & k_{(t-1)(t-1)} & 0_{(t-2)(t-1)} & 0_{(t-2)t} & & \\ & & & 0_{t(t-1)} & \kappa_{(t-1)(t-1)} & 0_{(t-1)t} & & & \\ & & & & k_{tt} & & & & \end{array} \right\rangle : \det([k_{ij}]) = 1 = \det([k_{lk}]) \right\},$$

where $k_{ij}, k_{lk} \in F, 1 \le i, j \le t,$

$1 \le l, k \le t-1;\ t = \dfrac{n+1}{2}\ ,\ n \in 2Z^{+} + 1.$

Since $\det(I_n) = 1,\ I_n$ belongs to $SKR_n(F).$ If A_n and B_n belong to $SKR_n(F),$ then,

$\det(A_n \circ B_n) = \det(A_n) \circ \det(B_n) = (1) \circ (1) = 1,$

$\Rightarrow A_n \circ B_n \in SKR_n(F).$

Also, $\det(A_n^{-1}) = \dfrac{1}{\det(A_n)} = \dfrac{1}{1} = 1.$

$\Rightarrow A_n^{-1} \in SKR_n(F).$ Thus, associativity law holds in $SKR_n(F)$ as in $SR_n(F).$ So $SKR_n(F)$ is a subgroup of $SR_n(F).$

Now, for any special rhotrix $X_n \in SR_n(F)$ and any special scalar rhotrix $A_n \in SKR_n(F),$ we have $\det(X_n^{-1} \circ A_n \circ X_n) = 1.$ Hence, $X_n^{-1} \circ A_n \circ X_n$ belongs to $SKR_n(F).$ So, $SKR_n(F)$ is a special scalar normal subgroup of $SR_n(F).$

Finally, to show that the cosets of $SKR_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets is as follows: Let the operation of coset multiplication in $SKR_n(F)$ be defined by

$\left(A_n \circ SKR_n(F)\right) \circ \left(B_n \circ SKR_n(F)\right)$
$= (A_n \circ B_n) \circ SKR_n(F)$

Then the rhotrix coset multiplication is well –defined, since

$\left(A_n \circ SKR_n(F)\right) \circ \left(B_n \circ SKR_n(F)\right)$
$= A_n \circ \left(SKR_n(F) \circ B_n\right) \circ SKR_n(F)$
$= A_n \circ \left(B_n \circ SKR_n(F)\right) \circ SKR_n(F)$
$= (A_n \circ B_n) \circ \left(SKR_n(F) \circ SKR_n(F)\right)$
$= (A_n \circ B_n) \circ SKR_n(F)$

using the fact that $SKR_n(F)$ is a special scalar normal subgroup of $SR_n(F),$ so

$SKR_n(F) \circ B_n = B_n \circ SKR_n(F)$ and

$\left(SKR_n(F) \circ SKR_n(F)\right) = SKR_n(F)$.

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $SR_n(F).$ Notice that $SKR_n(F)$ is the identity element of $SR_n(F)/SKR_n(F)$, since

$\left(A_n \circ SKR_n(F)\right) \circ SKR_n(F)$
$= A_n \circ \left(SKR_n(F) \circ SKR_n(F)\right) = A_n \circ SKR_n(F)$

and

$SKR_n(F) \circ \left(A_n \circ SKR_n(F)\right)$
$= \left(SKR_n(F) \circ A_n\right) \circ SKR_n(F)$
$= \left(A_n \circ SKR_n(F)\right) \circ SKR_n(F) = A_n \circ SKR_n(F)$

Lastly, $A_n^{-1} \circ SKR_n(F)$ is the inverse of $A_n \circ SKR_n(F),$ since

$\left(A_n^{-1} \circ SKR_n(F)\right) \circ \left(A_n \circ SKR_n(F)\right)$
$= \left(A_n^{-1} \circ A_n\right) \circ SKR_n(F) = I_n \circ SKR_n(F) = SKR_n(F)$

and

$\left(A_n \circ SKR_n(F)\right) \circ \left(A_n^{-1} \circ SKR_n(F)\right)$
$= \left(A_n \circ A_n^{-1}\right) \circ SKR_n(F) = I_n \circ SKR_n(F) = SKR_n(F)$

Hence, $SR_n(F)/SKR_n(F)$ is a quotient group under operation of coset multiplication.

Theorem 9

Let $(SR_n(F), \circ)$ be the non-commutative special rhotrix group and let $I_n(F)$ be a subset of $SR_n(F),$ consisting of identity rhotrix which has determinant as 1, then $I_n(F)$ is a trivial normal subgroup of $SR_n(F)$ and there exist the quotient group $SR_n(F)/I_n(F)$ under operation of coset multiplication, defined by

$\left(A_n \circ I_n(F)\right) \circ \left(B_n \circ I_n(F)\right) = (A_n \circ B_n) \circ I_n(F).$

Proof

First, it will be shown that $I_n(F)$ is a normal subgroup of $SR_n(F)$ and finally, show that the cosets of $I_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets.

Let

$$I_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & 1_{11} & & & & \\ & & 0_{21} & 1_{11} & 0_{12} & & & \\ & 0_{31} & 0_{21} & 1_{22} & 0_{12} & 0_{13} & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0_{t1} \ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0_{1t} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & 0_{t(t-2)} & 0_{(t-1)(t-2)} & 1_{(t-1)(t-1)} & 0_{(t-2)(t-1)} & 0_{(t-2)t} & & \\ & & 0_{t(t-1)} & 1_{(t-1)(t-1)} & 0_{(t-1)t} & & & \\ & & & 1_{tt} & & & & \end{array} \right\rangle, \right.$$
$$\left. : a_{ij}, c_{lk} \in F, \det([1_{ij}]) = 1 = \det([1_{lk}]) \right\}$$

where $1 \le i, j \le t,$

$$1 \le l, k \le t-1;\ t = \frac{n+1}{2}\ ,\ n \in 2Z^+ + 1.$$

Since $\det(I_n) = 1,\ I_n$ belongs to $I_n(F).$ If A_n and B_n belong to $I_n(F),$ then,

$$\det(A_n \circ B_n) = \det(A_n) \circ \det(B_n) = (1) \circ (1) = 1,$$
$$\Rightarrow A_n \circ B_n \in I_n(F).$$

Also, $\det(A_n^{-1}) = \dfrac{1}{\det(A_n)} = \dfrac{1}{1} = 1.$

$\Rightarrow A_n^{-1} \in I_n(F).$ Associativity holds in $I_n(F)$ as in $SR_n(F).$ Thus, $I_n(F)$ is a subgroup of $SR_n(F).$

Now, for any special rhotrix $X_n \in SR_n(F)$ and any identity rhotrix $A_n \in I_n(F),$ we have $\det(X_n^{-1} \circ A_n \circ X_n) = 1.$ Hence, $X_n^{-1} \circ A_n \circ X_n$ belongs to $I_n(F).$ So, $I_n(F)$ is a trivial normal subgroup of $SR_n(F).$

Finally, to show that the cosets of $I_n(F)$ in $SR_n(F)$ forms a quotient group under multiplication of cosets is as follows:

Let the operation of coset multiplication in $I_n(F)$ be defined by

$$\left(A_n \circ I_n(F)\right) \circ \left(B_n \circ I_n(F)\right) = (A_n \circ B_n) \circ I_n(F)$$

Then the rhotrix coset multiplication is well –defined, since

$$\left(A_n \circ I_n(F)\right) \circ \left(B_n \circ I_n(F)\right)$$
$$= A_n \circ \left(I_n(F) \circ B_n\right) \circ I_n(F)$$
$$= A_n \circ \left(B_n \circ I_n(F)\right) \circ I_n(F)$$
$$= (A_n \circ B_n) \circ \left(I_n(F) \circ I_n(F)\right)$$
$$= (A_n \circ B_n) \circ I_n(F)$$

using the fact that $I_n(F)$ is a trivial normal subgroup of $SR_n(F),$ so

$I_n(F) \circ B_n = B_n \circ I_n(F)$ and

$\left(I_n(F) \circ I_n(F)\right) = I_n(F).$

Associativity of rhotrix coset multiplication follows from the fact that associativity holds in $SR_n(F).$ Notice that $I_n(F)$ is the identity element of $SR_n(F)/I_n(F),$ since

$$\left(A_n \circ I_n(F)\right) \circ I_n(F)$$
$$= A_n \circ \left(I_n(F) \circ I_n(F)\right) = A_n \circ I_n(F)$$

and

$$I_n(F) \circ \left(A_n \circ I_n(F)\right) = \left(I_n(F) \circ A_n\right) \circ I_n(F)$$
$$= \left(A_n \circ I_n(F)\right) \circ I_n(F) = A_n \circ I_n(F)$$

Lastly, $A_n^{-1} \circ I_n(F)$ is the inverse of $A_n \circ I_n(F),$ since

$$\left(A_n^{-1} \circ I_n(F)\right) \circ \left(A_n \circ I_n(F)\right)$$
$$= \left(A_n^{-1} \circ A_n\right) \circ I_n(F) = I_n \circ I_n(F) = I_n(F)$$

and

$$\left(A_n \circ I_n(F)\right) \circ \left(A_n^{-1} \circ I_n(F)\right)$$
$$= \left(A_n \circ A_n^{-1}\right) \circ I_n(F) = I_n \circ I_n(F) = I_n(F)$$

Hence, $SR_n(F)/I_n(F)$ is a quotient group under operation of coset multiplication.

Remark

It will be interesting to notice the following compositions of subnormal series:

(i) $\{I\} \rhd\rhd (SDR_n(F), \circ) \rhd\rhd (SRTR_n(F), \circ) \rhd\rhd (SR_n(F), \circ) \rhd\rhd (GR_n(F), \circ)$

(ii) $\{I\} \rhd\rhd (SDR_n(F), \circ) \rhd\rhd (SLTR_n(F), \circ) \rhd\rhd (SR_n(F), \circ) \rhd\rhd (GR_n(F), \circ)$

(iii) $(GR_n(F), \circ)/\{I\} \rhd\rhd (GR_n(F), \circ)/(SDR_n(F), \circ) \rhd\rhd (GR_n(F), \circ)/(SLTR_n(F), \circ) \rhd\rhd (GR_n(F), \circ)/(SR_n(F), \circ)$

(iv) $$(GR_n(F),\circ)/\{I\} \rhd\rhd (GR_n(F),\circ)/(SDR_n(F),\circ) \rhd\rhd (GR_n(F),\circ)/(SRTR_n(F),\circ) \rhd\rhd (GR_n(F),\circ)/(SR_n(F),\circ)$$

Example

Consider the finite rhotrix group

$$FGR_3(Z_2) = \left\{ \begin{array}{l} R1 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle, R2 = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle, R3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle, \\ R4 = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle, R5 = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle, R6 = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle \end{array} \right\}$$

recorded from Mohammed and Okon(2016).

Notice that the elements (rhotrices) of this group draw entries from prime field $Z_2 = \{0,1\}$. Then, the group

$$NFGR_3(Z_2) = \left\{ R1 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle, R2 = \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle, R4 = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle \right\}$$

whose table appears below, is a special normal subgroup of $FGR_3(Z_2)$.

$\circ$	$R1$	$R2$	$R4$
$R1$	$R1$	$R2$	$R4$
$R2$	$R2$	$R4$	$R1$
$R3$	$R3$	$R5$	$R6$
$R4$	$R4$	$R1$	$R2$
$R5$	$R5$	$R6$	$R3$
$R6$	$R6$	$R3$	$R5$

The distinct cosets of $NFGR_3(Z_2)$ in $FGR_3(Z_2)$ are $\{R1, R2, R4\}$ and $\{R3, R5, R6\}$, these cosets are the elements of the quotient group $FGR(Z_2)/NFGR(Z_2)$ possessing the group table below.

$\circ$	$\{R1,R2,R4\}$	$\{R3,R5,R6\}$
$\{R1,R2,R4\}$	$\{R1,R2,R4\}$	$\{R3,R5,R6\}$
$\{R3,R5,R6\}$	$\{R3,R5,R6\}$	$\{R1,R2,R4\}$

Conclusion

A presentation of the concept of normal subgroups and quotient groups having rhotrix set as underlying set. Concrete examples had also been given in this work, so that it can further serve the purpose of reducing abstractions during teaching and learning of these concepts in abstract algebra. In the future, it may be interesting to consider extension of isomorphism theorems to subgroups of non-commutative general rhotrix group of size n over an arbitrary field F.

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