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RELATIONS AND FUNCTIONS IN SOFT MULTISET CONTEXT

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ABSTRACT
After recalling the concepts of soft sets, multisets and soft multisets, soft multiset relations and functions, together with some of their properties such as injective, surjective and bijective functions are introduced. It is further shown that some properties holding in multisets does not hold in soft multisets. Some related results are also established.

Keywords: Soft set, Multiset, Soft Multiset, Relations, Functions.

1. Introduction

Soft set which is a mapping from a set of parameters to a power set of a universe was initiated with the aim of modeling uncertainty in real life situation. The theory has applications in many areas such as decision making, medical diagnosis, data analysis, forecasting, game theory etc. to mention a few, as shown by (Maji *et al.*, 2002), (Zou and Xiao, 2002), (Majumdar and Samanta, 2008), (Feng *et al.*, 2010) and (Atmaca and Zorlutuna, 2013).

Multiset (mset, for short) which is an unordered collection of objects where unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted is initiated with the aim of addressing repetition which is significant in real life situations. Multisets are being extensively used in mathematics, theoretical computer science, biosystems (such as membrane computing and DNA computing), economics, formal language theory, social sciences and so on (see [(Knuth, 1973), (Knuth, 1981), (Blizard, 1991), (Eilenberg, 1994), (Singh *et al.*, 2007), (Singh and Isah, 2015) and (Isah and Tella, 2015)] for example).

Soft multiset was defined by (Alkhazalah *et al.*, 2011), (Majundar, 2012), (Babitha and Sunil, 2013) and (Osmanoglu and Tokat, 2013) in different ways. As multisets are generalization of sets (Blizard, 1989), that of (Osmanoglu and Tokat, 2013) serves as a generalization of soft sets, thus adopted. In this paper Soft multiset relations are introduced as a sub soft multiset of the Cartesian product of the soft multisets and many related concepts such as functions and some of its properties are discussed. Some related results are also presented.

2.1 Soft set

Definition 2.1.1 (Molodtsov, 1999), (Sezgin and Atagan, 2011)
Let U be an initial universe set and E a set of parameters or attributes with respect to U. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (F,A) or F_A is called a *soft set* over U, where F is a mapping given by $F: A \longrightarrow P(U)$.
In other words, a soft set (F,A) over U is a parameterized family of subsets of U. For $e \in A,\ F(A)$ may be considered as the set of e-elements or e-approximate elements of the soft set (F,A). Thus (F,A) is defined as

$$(F,A) = \{F(e) \in P(U): e \in E, F(e) = \emptyset \text{ if } e \notin A\}.$$

2.2 Multisets

Definition 2.2.1 (Jena and Ghosh, 2001)
An mset M drawn from the set X is represented by a function *Count* M or C_M defined as $C_M: X \longrightarrow \mathbb{N}$.
Let M be a multiset from X with x appearing n times in M. It is denoted by $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, \dots, k_n/x_n\}$ where M is a multiset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on.

Definitions 2.2.2 (Jena and Ghosh, 2001)
Let M and N be two msets drawn from a set X. Then
(a) $M \subseteq N$ iff $C_M(x) \leq C_N(x)$ for all $x \in X$. Clearly, every element of M is an element of N.
(b) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
(c) $M \cup N = max\{C_M(x), C_N(x)\}$ for all $x \in X$.
(d) $M \cap N = min\{C_M(x), C_N(x)\}$ for all $x \in X$.
(e) $M - N = max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$.

Definition 2.2.3 (Girish and Sunil, 2009)
Let M be a multiset drawn from a set X. The support set of M denoted by M^* is a subset of X given by $M^* = \{x \in X: C_M(x) > 0\}$. Note that $M \subseteq N$ iff $M^* \subseteq N^*$.
The power multiset of a given mset M, denoted by $P(M)$ is the multiset of all submultisets of M, and the power set of a multiset M is the support set of $P(M)$, denoted by $P^*(M)$.

Example 2.2.4 let $M = \{2/x, 2/y\}$, then $M^* = \{x, y\}$ and $P(M) = \{\emptyset, 2/\{1/x\}, 2/\{1/y\}, \{2/x\}, \{2/y\}, 4/\{1/x, 1/y\}, 2/\{2/x, 1/y\}, 2/\{1/x, 2/y\}, \{2/x, 2/y\}$. Of course $P^*(M) = \{\emptyset, \{1/x\}, \{1/y\}, \{2/x\}, \{2/y\}, \{1/x, 1/y\}, \{2/x, 1/y\}, \{1/x, 2/y\}, \{2/x, 2/y\}$.

Definitions 2.2.5 (Girish and Sunil, 2009)
Let M_1 and M_2 be msets drawn from a set X. Then

(a) The Cartesian product of M_1 and M_2 is defined as
$M_1 \times M_2 = \{(m/x, n/y)/mn: x \in^m M_1, y \in^n M_2\}$.
The Cartesian product of three or more nonempty msets is defined by generalizing the definition of the Cartesian product of two msets.

(b) A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x,y)$ and $C_2(x,y)$. We denote m/x related to n/y by $m/x\ R\ n/y$.

(c) The Domain and Range of the mset relation R on M is defined as follows:

$Dom\ R = \{ x \in^r M : \exists y \in^s M$ such that $r/x\ R\ s/y\}$ where $C_{DomR}(x) = \sup\{C_1(x,y) : x \in^r M\}$.
$Ran\ R = \{y \in^s M : \exists x \in^r M$ such that $r/x\ R\ s/y\}$ where $C_{RanR}(y) = \sup\{C_2(x,y) : y \in^s M\}$.

(d) An mset relation f is called an mset function if for every element m/x in Dom f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring only $C_1(x,y)$ times. For functions between arbitrary msets it is essential that images of indistinguishable elements of the domain must be indistinguishable elements of the range but the images of the distinct elements of the domain need not be distinct elements of the range.

2.3 Soft Multiset (Soft mset, for short)

Definition 2.3.1 (Osmanoglu and Tokat, 2013)
Let U be a universal multiset, E be a set of parameters and $A \subseteq E$. Then a pair (F,A) or F_A is called a soft multiset where F is a mapping given by $F : A \longrightarrow P^*(U)$. For all $e \in A$, the mset $F(e)$ is represented by a count function $C_{F(e)}: U^* \longrightarrow \mathbb{N}$.

Example 2.3.2 Let the universal mset $U = \{3/w, 2/x, 4/y, 1/z\}$, the parameter set $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, $A = \{e_1, e_2, e_3\}$ and the mapping $F : A \longrightarrow P^*(U)$ be defined as $F(e_1) = \{2/w, 1/y, 1/z\}, F(e_2) = \{1/w, 2/x, 3/y\}$ and $F(e_3) = \{1/x, 2/y\}$. That is, (F,A) is a soft multiset such that for all $e \in A$, the multiset $F(e)$ is represented by a count function $C_{F(e)}: U^* \longrightarrow \mathbb{N}$ as

$$C_{F(e_1)}(w) = 2,\quad C_{F(e_1)}(x) = 0,\quad C_{F(e_1)}(y) = 1,\quad C_{F(e_1)}(z) = 1$$
$$C_{F(e_2)}(w) = 1,\quad C_{F(e_2)}(x) = 2,\quad C_{F(e_2)}(y) = 3,\quad C_{F(e_2)}(z) = 0$$
$$C_{F(e_3)}(w) = 0,\quad C_{F(e_3)}(x) = 1,\quad C_{F(e_3)}(y) = 2,\quad C_{F(e_3)}(z) = 0$$

Thus, $(F,A) = \{ (e_1, \{2/w, 1/y, 1/z\}), (e_2, \{1/w, 2/x, 3/y\}), (e_3, \{1/x, 2/y\})\}$.

Definition 2.3.3 (Osmanoglu and Tokat, 2013)
Let (F,A) and (G,B) be two soft multisets over U. Then
(a) (F,A) is a soft submultiset of (G,B) written $(F,A) \sqsubset (G,B)$ if
i. $A \subseteq B$
ii. $C_{F(e)}(x) \le C_{G(e)}(x), \forall x \in U^*, \forall e \in A$.

$(F,A) = (G,B) \Leftrightarrow (F,A) \sqsubset (G,B)$ and $(G,B) \sqsubset (F,A)$.
Also, if $(F,A) \sqsubset (G,B)$ and $(F,A) \neq (G,B)$ then (F,A) is called a proper soft submset of (G,B) and (F,A) is a whole soft submset of (G,B) if $C_{F(e)}(x) = C_{G(e)}(x), \forall x \in F(e)$.

Example 2.3.4
Considering (F,A) in Example 2.3.2, if $B = \{e_1, e_2, e_3\}$ and $C = \{e_1, e_3\}$, then
$(G,B) = \{(e_1, \{2/w, 1/z\}), (e_2, \{1/w, 2/x, 2/y\}), (e_3, \{1/x, 2/y\})\}$ is a proper soft submultiset of (F,A), and
$(H,C) = \{(e_1, \{2/w, 1/y, 1/z\}), (e_3, \{1/x, 2/y\})\}$ is a whole soft submultiset of (F,A).

(b) **Union:**
$(F,A) \sqcup (G,B) = (H,C)$ where $C = A \cup B$ and $C_{H(e)}(x) = max\{C_{F(e)}(x), C_{G(e)}(x)\}, \forall e \in C, \forall x \in U^*$.

(c) **Intersection:**
$(F,A) \sqcap (G,B) = (H,C)$ where $C = A \cap B$ and $C_{H(e)}(x) = min\{C_{F(e)}(x), C_{G(e)}(x)\}, \forall e \in C, \forall x \in U^*$.

(d) **Difference:** $(F,E)\backslash(G,E) = (H,E)$ where $C_{H(e)}(x) = max\{C_{F(e)}(x) - C_{G(e)}(x), 0\}, \forall x \in U^*$.

(e) **Null:** A soft multiset (F,A) is called a Null soft multiset denoted by $\emptyset$ if $\forall e \in A, F(e) = \emptyset$.

Complement The complement of a soft multiset (F,A), denoted by $(F,A)^c$, is defined by $(F,A)^c = (F^c, A)$ where $F^c: A \longrightarrow P^*(U)$ is a mapping given by $F^c(e) = U\backslash F(e), \forall e \in A$ where $C_{F^c(e)}(x) = C_U(x) - C_{F(e)}(x),\ \forall x \in U^*$.

Remark 2.3.5
The property of submset does not hold for soft submset. For example, let (*F*, *A*) and (*G*, *B*) be soft multisets over a universe $U = \{3/w, 7/x, 9/y, 8/z\}$ and let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, $A = \{e_1, e_2\}$, $B = \{e_1, e_2, e_3\}$, $F(e_1) = \{2/x, 3/y\}, F(e_2) = \{1/x, 2/z\}, G(e_1) = \{2/x, 3/y\}, G(e_2) = \{2/x, 1/y, 3/z\}$, $G(e_3) = \{2/w, 5/x, 3/y\}$, then
$A \subseteq B$ and $C_{F(e)}(x) \le C_{G(e)}(x), \forall x \in U^*, \forall e \in A$, thus $(F,A) \sqsubset (G,B)$. However, $\big(e_2, F(e_2)\big) \in (F,A)$ but $\big(e_2, F(e_2)\big) \notin (G,B)$.

Definition 2.3.6 Let (F,A) and (G,B) be soft multisets over a universe U, then
(a) (F,A) OR (G,B) denoted by $(F,A) \vee (G,B)$ is the soft multisets $(H, A \times B)$ where $H(a,b) = F(a) \cup G(b)$.

(b) (F,A) AND (G,B) denoted by $(F,A) \wedge (G,B)$ is the soft multisets $(K, A \times B)$ where $K(a,b) = F(a) \cap G(b)$.

(c) $((F,A) \vee (G,B))^c = (H, A \times B)^c = (H^c, A \times B)$.

(d) $((F,A) \wedge (G,B))^c = (K, A \times B)^c = (K^c, A \times B)$.

Example 2.3.7 Let $U = \{3/w, 7/x, 9/y, 8/z\}, E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}, A = \{e_1, e_2, e_3\}$ and $B = \{e_3, e_4, e_5\}$. Suppose further that $(F,A) = \{(e_1, \{2/x, 3/y\}), (e_2, \{1/x, 2/z\}), (e_3, \{2/x, 3/y\})\}$ and $(G,B) = \{(e_3, \{2/w, 3/y\}), (e_4, \{3/x, 2/z\}), (e_5, \{1/y, 4/z\})\}$, then

$$(F,A) \vee (G,B) = (H, A \times B)$$
$$= \{H(e_1,e_3), H(e_1,e_4), H(e_1,e_5), H(e_2,e_3), H(e_2,e_4), H(e_2,e_5), H(e_3,e_3), H(e_3,e_4), H(e_3,e_5)\} =$$
$$\{((e_1,e_3), \{2/w, 2/x, 3/y\}),$$
$$((e_1,e_4), \{3/x, 3/y, 2/z\}),$$
$$((e_1,e_5), \{2/x, 3/y, 4/z\}),$$
$$((e_2,e_3), \{2/w, 1/x, 3/y, 2/z\}),$$
$$((e_2,e_4), \{3/x, 2/z\}),$$
$$((e_2,e_5), \{1/x, 1/y, 4/z\}),$$
$$((e_3,e_3), \{2/w, 2/x, 3/y\}),$$
$$((e_3,e_4), \{3/x, 3/y, 2/z\}),$$

$((e_3, e_5), \{2/x, 3/y, 4/z\})\}$.

$(F, A) \wedge (G, B) =$
$\{((e_1, e_3), \{3/y\}), ((e_1, e_4), \{2/x\}),$
$((e_1, e_5), \{1/y\}), ((e_2, e_3), \emptyset),$
$((e_2, e_4), \{1/x, 2/z\}), ((e_2, e_5), \{2/z\}),$
$((e_3, e_3), \{3/y\}), ((e_3, e_4), \{2/x\}),$
$((e_3, e_5), \{1/y\})\}$.

$\big((F, A) \vee (G, B)\big)^c = (H^c, A \times B)$
$= \{H^c(e_1, e_3), H^c(e_1, e_4), H^c(e_1, e_5),$
$H^c(e_2, e_3), H^c(e_2, e_4), H^c(e_2, e_5), H^c(e_3, e_3),$
$H^c(e_3, e_4), H^c(e_3, e_5)\} =$
$\{((e_1, e_3), \{1/w, 4/x, 6/y, 8/z\}),$
$((e_1, e_4), \{3/w, 4/x, 6/y, 6/z\}),$
$((e_1, e_5), \{3/w, 5/x, 6/y, 4/z\}),$
$((e_2, e_3), \{1/w, 6/x, 6/y, 6/z\}),$
$((e_2, e_4), \{3/w, 4/x, 9/y, 6/z\}),$
$((e_2, e_5), \{3/w, 6/x, 8/y, 4/z\}),$
$((e_3, e_3), \{1/w, 5/x, 6/y, 8/z\}),$
$((e_3, e_4), \{3/w, 4/x, 6/y, 6/z\}),$
$((e_3, e_5), \{3/w, 5/x, 6/y, 4/z\})\}$.

$((F, A) \wedge (G, B))^c = (K^c, A \times B) =$
$\{((e_1, e_3), \{3/w, 7/x, 6/y, 8/z\}),$
$((e_1, e_4), \{3/w, 5/x, 9/y, 8/z\}),$
$((e_1, e_5), \{3/w, 7/x, 8/y, 8/z\}),$
$((e_2, e_3), \{3/w, 7/x, 9/y, 8/z\}),$
$((e_2, e_4), \{3/w, 6/x, 9/y, 6/z\}),$
$((e_2, e_5), \{3/w, 7/x, 9/y, 6/z\}),$
$((e_3, e_3), \{3/w, 7/x, 6/y, 8/z\}),$
$((e_3, e_4), \{3/w, 5/x, 9/y, 8/z\}),$
$((e_3, e_5), \{3/w, 7/x, 8/y, 8/z\})\}$.

3. Soft multisets Relations and Functions

Definition 3.1 Cartesian product
Let (F, A) and (G, B) be two soft multisets over U, then the Cartesian product of (F, A) and (G, B) is defined as $(F, A) \times (G, B) = (K, A \times B)$ where $K: A \times B \longrightarrow P^*(U \times U)$ and $K(a, b) = F(a) \times G(b)$ for $(a, b) \in A \times B$ and $m/x \in F(a) \wedge n/y \in G(b) \Rightarrow m/x \times n/y = (m/x, n/y)/mn$.
That is, $K(a, b) = \{(m/k_i, n/k_j)/mn : m/k_i \in F(a) \text{ and } n/k_j \in G(b)\}$

The Cartesian product of three or more nonempty soft multisets can be defined by generalizing the definition of the Cartesian product of two soft multisets. The Cartesian product of n nonempty soft multisets $(F_1, A_1), (F_2, A_2), \ldots, (F_n, A_n)$ is the soft multisets of all ordered $n-$tuples $(k_1, k_2, k_3, \ldots, k_n)$ where $k_i \in F_i(a_i)$

Example 3.2
Let $U = \{5/p, 7/q, 3/w, 2/x, 4/y, 1/z\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, $A = \{e_1, e_2, e_3\}$, $B = \{e_5, e_7\}$ and $F(e_1) = \{2/p, 1/x, 3/y\}$, $F(e_2) = \{1/w, 2/x\}$, $F(e_3) = \{2/y\}$, $G(e_5) = \{2/y, 1/z\}$, $G(e_7) = \{2/w, 1/y\}$. Then
$(F, A) \times (G, B) = (K, A \times B) =$
$\{K(e_1, e_5), K(e_1, e_7), K(e_2, e_5), K(e_2, e_7),$
$K(e_3, e_5), K(e_3, e_7)\} =$
$= \{((e_1, e_5), \{(2/p, 2/y)/4, (2/p, 1/z)/2,$
$(1/x, 2/y)/2, (1/x, 1/z)/1, (3/y, 2/y)/6,$
$(3/y, 1/z)/3\}), ((e_1, e_7), \{(2/p, 2/w)/4,$
$(2/p, 1/y)/2, (1/x, 2/w)/2, (1/x, 1/y)/1,$
$(3/y, 2/w)/6, (3/y, 1/y)/3\}),$
$((e_2, e_5), \{(1/w, 2/y)/2, (1/w, 1/z)/1,$
$(2/x, 2/y)/4, (2/x, 1/z)/2\}),$
$((e_2, e_7), \{(1/w, 2/w)/2, (1/w, 1/y)/1,$
$(2/x, 2/w)/4, (2/x, 1/y)/2\}),$
$((e_3, e_5), \{(2/y, 2/y)/4, (2/y, 1/z)/2\}),$
$((e_3, e_7), \{(2/y, 2/w)/4, (2/y, 1/y)/2\})\}$.

Definition 3.3 Soft multiset Relation
Let (F, A) and (G, B) be two soft msets over U, then a relation from (F, A) to (G, B) is a soft submset of $(F, A) \times (G, B)$ i.e., it is a soft multiset (H_1, C) where $C \subset A \times B$ and $H_1 = H(a, b)$ for all $(a, b) \in C$. Moreover, a soft submset of $(F, A) \times (F, A)$ is called a relation on (F, A).
Equivalently, in the parameterized form, if $(F, A) = \{F(a), F(b), \ldots\}$ then $F(a)RF(b)$ iff
$F(a) \times F(b) \in R$.

Definition 3.4
Let R be a soft multiset relation from (F, A) to (G, B). Then (dom R) the domain of R is defined as the soft mset (DF, A_1), such that $A_1 \subseteq A$, $A_1 = a \in A: H(a, b) \in R$ for some $b \in B\}$, and $DF(a_1) = F(a)$, for all $a \in A_1$ where $C_{DomR}(a) = Sup\{C_1(a, b) \mid a \in DF(a_1)$.
The range of R (ran R) is defined as the soft multiset (RG, B_1), where $B_1 \subseteq B$,
$B_1 = b \in B: H(a, b) \in R$ for some $a \in A\}$ and $RG(b_1) = G(b)$, for all $b \in B_1$ where $C_{RanR}(b) = Sup\{C_2(a, b) \mid b \in RG(b_1)$.

Example 3.5
Consider a universe U consisting of students offering course q, course w, course x and course y with their multiplicities. The parameters A having brilliant, average and dull, and B having healthy, active, inactive and physically disable presented as $U = \{7/q, 3/w, 2/x, 4/y\}$, $A = \{e_1, e_2, e_3\}$, $B = \{e_4, e_5, e_6, e_7\}$ and
$F(e_1) = \{3/q, 1/x, 2/y\}$, $F(e_2) = \{2/w, 2/x, 1/y\}$, $F(e_3) = \{4/q, 1/w, 2/y\}$
$G(e_4) = \{3/q, 2/x, 2/y\}$, $G(e_5) = \{2/w, 1/y\}$, $G(e_6) = \{4/q, 1/w\}$, $G(e_7) = \{2/w, 2/x, 2/y\}$
Let us define a relation R from (F, A) to (G, B) as $F(a)RG(b)$ iff $F(a) \supseteq G(b)$.
Then $R = \{F(e_2) \times G(e_5), F(e_3) \times G(e_6)\}$
$= \{((e_2 \times e_5), \{\{2/w, 2/x, 1/y\} \times \{2/w, 1/y\}\}), ((e_3 \times e_6), \{\{4/q, 1/w, 2/y\} \times \{4/q, 1/w\}\})\}$
$= \{((e_2 \times e_5), \{(2/w, 2/w)/4, (2/w, 1/y)/2, (2/x, 2/w)/4, (2/x, 1/y)/2, (1/y, 2/w)/2, (1/y, 1/y)/1\}), ((e_3 \times e_6), \{(4/q, 4/q)/16, (4/q, 1/w)/4, (1/w, 4/q)/4, (1/w, 1/w)/1, (2/y, 4/q)/8, (2/y, 1/w)/2\})\}$.
Dom $R = \{((e_2 \times e_5), \{2/w, 2/x, 1/y\}),$
$((e_3 \times e_6), \{1/w, 2/y, 4/q\})\}$.
Ran $R = \{((e_2 \times e_5), \{2/w, 1/y\}), ((e_3 \times e_6), \{4/q, 1/w\})\}$.

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Definition 3.6
Let (F, A) be a soft multiset defined on the attribute set A and R be a relation defined on A i.e., $R \subset A \times A$. Then the induced soft multiset relation R_A on (F, A) is defined as
$F(a)R_AF(b)$ iff aRb, for $a, b \in A$.

Example 3.7
Let consider a soft multiset (F, A) over a universal multiset U of people attending a routine exercise in a gymnasium with p-married men, q-unmarried men, w-married women, x-unmarried women, and y-others, and A the set of their qualifications, where $U = \{9/p, 5/q, 3/w, 7/x, 6/y\}$, $A = \{BSc., B.A., B.Tech, MSc., M.A., M.Tech\} = \{b_1, b_2, b_3, m_1, m_2, m_3\}$.
Define a relation R on A as aRb iff a and b are bachelor's degree. Then R_A on (F, A) is
$\{F(b_1) \times F(b_2), F(b_2) \times F(b_1), F(b_1) \times F(b_3), F(b_3) \times F(b_1), F(b_2) \times F(b_3), F(b_3) \times F(b_2), F(b_1) \times F(b_1), F(b_2) \times F(b_2), F(b_3) \times F(b_3)\}$.

Definition 3.8 Soft Multiset Function
Let (F, A) and (G, B) be two non-empty soft multisets. Then a soft multiset relation f from (F, A) to (G, B) is called a *soft multiset function* if every element in the domain has a unique element in the range such that for every $K(a, b) \in f$, $(m/k_i, n/k_j)$ appears only m times for $m/k_i \in F(a)$ and $n/k_j \in G(b)$.

Example 3.9
Let $U = \{17/q, 8/w, 25/x, 7/y\}$, $A = \{e_1, e_2, e_3\}$, $B = \{e_4, e_5\}$ and
$F(e_1) = \{2/q, 3/x\}$, $F(e_2) = \{6/w, 5/x, 3/y\}$, $F(e_3) = \{2/q, 3/w, 4/y\}$
$G(e_4) = \{3/y, 4/x\}$, $G(e_5) = \{7/w, 5/y\}$ then $f = \{F(e_1) \times G(e_4), F(e_2) \times G(e_4), F(e_3) \times G(e_5)\}$ is a soft multiset function. Observe that
$F(e_1) \times G(e_4) = \{(e_1, e_4), (2/q, 3/y)/2, (2/q, 4/x)/2, (3/x, 3/y)/3, (3/x, 4/x)/3\}$.

Definition 3.10 Let (F, A) be a soft multiset over U. The support set or root set of (F, A) denoted by $(F, A)^* = (F^*, A)$ is a subset of U^* such that $(F^*, A) = \{x \in U^* : C_{F(a)}(x) > 0, \forall a \in A\}$.

Definition 3.11 Let $F(a) \in (F, A)$, then the support set or root set of $F(a)$ denoted by $F^*(a)$ is a subset of U^* such that $F^*(a) = \{x \in U^* : C_{F(a)}(x) > 0\}$.
In example 3.9, $(F^*, A) = \{q, x, w, y\}$ and $F^*(a) = \{q, x\}$.

Definition 3.12
A soft multiset function f from (F, A) to (G, B) is called *injective* (one-one) if
(i) $f: A \longrightarrow B$ is injective
(ii) $f: F^*(a) \longrightarrow G^*(b)$ is injective and
(iii) $C_{F(a)}(x) \leq C_{f(G(b))}(x), \forall x \in F^*(a)$

Definition 3.13
A soft multiset function f from (F, A) to (G, B) is called *surjective* (onto) if
(i) $f: A \longrightarrow B$ is surjective
(ii) $f: F^*(a) \longrightarrow G^*(b)$ is surjective and
(iii) $C_{F(a)}(x) \geq C_{f(G(b))}(x), \forall x \in F^*(a)$

Definition 3.14
A soft multiset function f from (F, A) to (G, B) is called *bijective* (one-one and onto) if
(i) $f: A \longrightarrow B$ is bijective
(ii) $f: F^*(a) \longrightarrow G^*(b)$ is bijective and
(iii) $C_{F(a)}(x) = C_{f(G(b))}(x), \forall x \in F^*(a)$

Example 3.15
Let $U = \{17/q, 8/w, 25/x, 7/y\}$, $A = \{e_1, e_2\}$, $B = \{e_4, e_5\}$ and
$F(e_1) = \{2/q, 3/x\}$, $F(e_2) = \{1/w, 3/y\}$, $G(e_4) = \{3/y, 4/x\}$, $G(e_5) = \{3/w, 4/y\}$ then
f from (F, A) to (G, B) given by
$f = \{F(e_1) \times G(e_4), F(e_2) \times G(e_5)\} = \{((e_1, e_4), \{(2/q, 3/y)/2, (2/q, 4/x)/2, (3/x, 3/y)/3, (3/x, 4/x)/3), ((e_2, e_5), \{(1/w, 3/w)/1, (1/w, 4/y)/1, (3/y, 3/w)/3, (3/y, 4/y)/3)\}$ is injective. However, f is not surjective since we have $(2/q, 3/y)/2, (1/w, 3/w)/1$ etc.
Moreover, if $A = \{e_3\}$, $B = \{e_7\}$ and $F(e_3) = \{2/q, 2/x\}$, $G(e_7) = \{2/w, 2/y\}$, then
$f = \{F(e_3) \times G(e_7)\} = \{((e_3, e_7), \{(2/q, 2/w)/2, (2/q, 2/y)/2, (2/x, 2/w)/2, (2/x, 2/y)/2)\}$ is bijective.

Definition 3.16
A constant soft multiset function f is a function in which every element of dom f has the same image. For example, let $A = \{e_3, e_6\}$, $B = \{e_7\}$ and $F(e_3) = \{2/q, 3/x\}, F(e_6) = \{2/w, 1/x\}, G(e_7) = \{1/w\}$, and
$f = \{F(e_3) \times G(e_7), F(e_6) \times G(e_7)\} = \{((e_3, e_7), \{(2/q, 1/w)/2, (3/x, 1/w)/3), ((e_6, e_7), \{(2/w, 1/w)/2, (1/x, 1/w)/1\}$ is a constant soft mset function.

Definition 3.17
The identity soft multiset function I on a soft multiset (F, A) is defined as a function $I: (F, A) \longrightarrow (F, A)$ as $I(F(a)) = F(a)$ for every $F(a) \in (F, A)$.

Theorem 3.18
Let $f: (F, A) \longrightarrow (G, B)$ be a soft multiset function and (H, C), (K, D) be soft submultisets of (F, A). Then
(a) If $(H, C) \subseteq (K, D) \Longrightarrow f(H, C) \subseteq f(K, D)$
(b) $f[(H, C) \cup (K, D)] = f(H, C) \cup f(K, D)$
(c) $f[(H, C) \cap (K, D)] \subseteq f(H, C) \cap f(K, D)$ equality holds if f is one to one.

Proof
(a) Consider any $G(b) \in (H, C)$.
Then $G(b) = f(F(a))$ for some $F(a)$ in (H, C)
Since $(H, C) \subseteq (K, D)$, $G(b) = f(F(a))$ for some $F(a)$ in (K, D)
$\Longrightarrow G(b) \in (K, D)$
Therefore $f(H, C) \subseteq f(K, D)$

(b) Let $G(b) \in f[(H, C) \cup (K, D)]$
Then $G(b) = f(F(a))$ for some $F(a)$ in $(H, C) \cup (K, D)$
$= f(F(a))$, for $F(a)$ in (H, C) or $F(a)$ in (K, D)
$\Longrightarrow G(b) \in f(H, C)$ or $f(K, D)$

$\Longrightarrow G(b) \in f(H,C) \cup f(K,D)$
Therefore $f[(H,C) \cup (K,D)] \subseteq f(H,C) \cup f(K,D)$

Since $(H,C) \subseteq (H,C) \cup (K,D)$ and $(K,D) \subseteq (H,C) \cup (K,D)$
$f(H,C) \subseteq f[(H,C) \cup (K,D)]$ and $f(K,D) \subseteq f[(H,C) \cup (K,D)]$
Thus, $f(H,C) \cup f(K,D) \subseteq f[(H,C) \cup (K,D)]$
Therefore $f[(H,C) \cup (K,D)] = f(H,C) \cup f(K,D)$
(c) Let $G(b) \in f[(H,C) \cap (K,D)]$
Then $G(b) = f(F(a))$ for some $F(a)$ in $(H,C) \cap (K,D)$
$= f(F(a))$, for $F(a)$ in (H,C) and $F(a)$ in (K,D)
$\Longrightarrow G(b) \in f(H,C)$ and $f(K,D)$
$\Longrightarrow G(b) \in f(H,C) \cap f(K,D)$
Therefore $f[(H,C) \cap (K,D)] \subseteq f(H,C) \cap f(K,D)$
Conversely, let $G(b) \in f(H,C) \cap f(K,D)$
$G(b) = f(H(c))$ for some $H(c) \in (H,C)$ and $G(b) = f(K(d))$ for some $K(d) \in (K,D)$
$\Longrightarrow G(b) = f(H(c)) = f(K(d))$
$\Longrightarrow H(c) = K(d)$ if f is one – one
$\Longrightarrow G(b) = f(H(c))$ for some $H(c) \in (H,C)$ and $H(c) \in (K,D)$
$\Longrightarrow G(b) = f(H(c))$ for some $H(c) \in (H,C) \cap (K,D)$
$\Longrightarrow G(b) \in f[(H,C) \cap (K,D)]$
Therefore $f(H,C) \cap f(K,D) \subseteq f[(H,C) \cap (K,D)]$
Thus, if f is one – one $f[(H,C) \cap (K,D)] = f(H,C) \cap f(K,D)$

Definition 3.19
A soft multiset function f is called *invertible* if its inverse soft multiset relation f^{-1} is a soft multiset function. That is, if $f: (F,A) \longrightarrow (G,B)$, then $f^{-1}: (G,B) \longrightarrow (F,A)$.
Example 3.20
The inverse of the soft multiset function $f = \{F(e_1) \times G(e_4), F(e_2) \times G(e_5)\}$ is $f^{-1} = \{G(e_4) \times F(e_1), G(e_5) \times F(e_2)\}$ and
$G(e_4) \times F(e_1) = \{(3/y, 2/q)/2, (4/x, 2/q)/2, (3/y, 3/x)/3, (4/x, 3/x)/3\}$. It is clear from this example that not every soft multiset function is invertible.

Theorem 3.21
Let $f: (F,A) \longrightarrow (G,B)$ be a soft mset function. Then
(a) $(f^{-1})^{-1}$ is the soft mset function f.
(b) $f^{-1}: (G,B) \longrightarrow (F,A)$ is a soft mset function if and only if f is bijective.
Proof
(a) Proof follows from definition.
(b) Let the contra positive statement be true i.e., f is not bijective if and only if f^{-1} is not a soft multiset function.
Suppose f is not bijective i.e., for distinct $F(a_1)$ and $F(a_2)$ in (F,A)
$f(F(a_1)) = G(b_1)$ and $f(F(a_2)) = G(b_1)$
i.e., $f^{-1}(b_1) = F(a_1)$ and $f^{-1}(b_1) = F(a_2)$
i.e., f^{-1} is not a soft mset function.
Conversely, suppose f^{-1} is not a soft multiset function, then there exists some $G(b_1) \in (G,B)$ such that $f^{-1}(b_1)$ contains at least two distinct elements $F(a_1)$ and $F(a_2)$ in (F,A). That is $f(F(a_1)) = G(b_1)$ and $f(F(a_2)) = G(b_1)$ which indicates that f is not bijective.

Conclusion
In this work, we carefully study the different definitions of soft multisets provided by various scholars and adopt the one that serves as the generalization of soft sets. After reviewing some concepts related to soft multisets, some operations on soft multisets were presented. It is further shown that some properties holding in multisets does not hold in soft multisets.

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