

NUMERICAL SOLUTION OF BLACK – SCHOLES PARTIAL DIFFERENTIAL EQUATION USING DIRECT SOLUTION OF SECOND - ORDER ORDINARY DIFFERENTIAL EQUATION WITH TWO - STEP HYBRID BLOCK METHOD OF ORDER SEVEN

¹Olaiya O. O., ²Oduwale H. K. and ³Odeyemi J. K.

¹Department of Mathematics, Nasarawa State University, Keffi /NMC ABUJA.

²Department of Mathematics, Nasarawa State University, Keffi.

³Department of Mathematics, University of Abuja.

*Corresponding Author Email Address: olaiya.o@gmail.com

ABSTRACT

This paper proposes a new numerical solution of Black-Scholes Partial Differential Equation using Direct solution of second-order Ordinary Differential Equation ODE with two-step hybrid Block Method of Order seven directly. The method is developed using interpolation and collocation techniques. The use of the power series approximate solution as an interpolation polynomial and its second derivative as a collocation equation is considered in deriving the method. Properties of the method such as zero stability, order, consistency, convergence and region of absolute stability are investigated. The new method is then applied to solve Black –Scholes equation after converting it to the system of second-order ordinary differential equations and the accuracy is better when compared with the existing methods in terms of error.

Keywords: single-step; hybrid block method; system of second order ordinary differential equations; collocation and Interpolation method; direct solution.

INTRODUCTION

In the historical backdrop of option pricing model, the Black-Mcholes or Black –Scholes –Merton model is a standout amongst the most generous model. This model shows the significance role that the mathematics plays in the field of finance .

The Black – Scholes model was first published by Black and Scholes (1973) in their seminar paper “The Pricing of options and corporate Liabilities” published in the Journal of political Economy .In the same year, they derived a partial differential equation, now called the Black –Scholes Equation, which estimates the price of the option over time .

Let us consider S as the price of the stock, which we consider as a random variable. $V(s, t)$ be the value of an option as a function of time and stock price, K , can be strike price, r be the risk –free interest rate, σ be the volatility/the standard deviation of the stock return ,and t be the time in years. Then the famous Black – Scholes equation that was developed by Fisher Black and Myron Scholes is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

The above equation is a second – order parabolic partial differential equation known as Black – Scholes equation ,is actually a variation of a famous equation in physics that models

the transfer of heat.

Simple Transformation of PDE to ODE

Black –Scholes equation is given by the following expression

$$\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial v}{\partial S} - rV = 0 \quad (1)$$

where $V(s,t)$ =price of option, σ is the volatility of stock price S , and V, t = period of time, r = interest rate (Company et al. , 2007). Firstly, it is expedient to transform this partial differential equation (PDE) into an ordinary differential equation (ODE) by proposing the following solution: $V(S, t) = V(S)e^{\lambda t}$.

Given that $\frac{\partial v}{\partial t} = V(S) * \lambda e^{\lambda t}$ and $\frac{\partial v}{\partial S} = \frac{\partial V(S)}{\partial S} e^{\lambda t}$, by substituting these equations into the PDE, we get

$$V(S)\lambda e^{\lambda t} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 V(S)}{dS^2} e^{\lambda t} + rS \frac{dV(S)}{dS} e^{\lambda t} - rV(S)e^{\lambda t} = 0 \quad (2)$$

The next step is to rearrange the equation to get second order ODE :

$$e^{\lambda t} \left[\frac{1}{2} \sigma^2 S^2 \frac{d^2 V(S)}{dS^2} + rS \frac{dV(S)}{dS} + V(S)(\lambda - r) \right] = 0. \quad (3)$$

The latter expression can be reduced to the following equation;

$$\frac{1}{2} \sigma^2 S^2 \frac{d^2 V(S)}{dS^2} + rS \frac{dV(S)}{dS} + V(S)(\lambda - r) = 0 \quad (4)$$

Where $e^{\lambda t} \neq 0$.

Ordinary differential equations (ODEs)

Ordinary differential equations (ODEs) are commonly used for mathematical modelling in many diverse fields such as engineering, operations research, industrial mathematics, behavioural sciences, artificial intelligence, management and sociology. Thus, mathematical modelling is the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provide insight, answers and guidance useful for the originating application. This type of problem can be formulated either in terms of first-order or higher-order ODEs.

In this Paper, the system of second – order ODEs of the following form is considered. We are interested in solving nonlinear time

dependant PDE of the form [Black – Scholes Equation].

$$\begin{aligned} {}^1y' &= {}^1f(x, {}^1y, {}^1y'), & {}^1y(x_0) &= a_0, & {}^1y'(x_0) &= b_0 \\ {}^2y'' &= {}^2f(x, {}^2y, {}^2y'), & {}^2y(x_0) &= a_1, & {}^2y'(x_0) &= b_1 \\ &\vdots & & \vdots & & \vdots \\ {}^m y'' &= {}^m f(x, {}^m y, {}^m y'), & {}^m y(x_0) &= a_{m-1}, & {}^m y'(x_0) &= b_{m-1} \end{aligned} \quad (5)$$

The method of solving higher-order ODEs by reducing them to a system of first-order equation involves more functions evaluation which to evaluate leads to computational burden as mentioned in (Master, 2011) and (Jator & Li., 2012). The multistep methods for solving higher –order ODEs directly have been developed by many scholars such as (Yusuph & Onumanyi, 2005). However, these researchers only applied their methods to solve single initial value problems of ODEs.

The aim of this paper is to develop a new numerical method for solving single second – order ODEs and systems of second-order ODEs directly.

Derivation of the method

We shall be considering a two–step hybrid block method with five off step point, $x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}$ and x_{n+2} for solving equation (5) is derived.

Let us consider the power series of the form

$${}^k y(x) = \sum_{i=0}^{r+s-1} a_i \left(\frac{x-x_n}{h}\right)^i, \quad k = 1, \dots, m \quad (6)$$

For $x \in [x_n, x_{n+1}]$ where $n = 1, 2, \dots, N - 1$, a_i are real coefficients, $r =$ the number of collocating points, $s =$ the number of interpolating points and $h = x_{n+1} - x_n$ is a constant step size of the partition of the interval $[a, b]$ which is given by $a < x < b; a = x_0 < x_1 < x_2 < \dots, x_{N-1} = b$
 Differentiating equation (6) twice give us

$$y'' = \sum_{i=0}^{r+s-1} \frac{i(i-1)a_i}{h^s} \left(\frac{x-x_n}{h}\right)^{i-2} \quad (7)$$

Equation (6) can be resolved in form of approximate solution thus;

$$\begin{aligned} y(x) &= \sum_{i=0}^{r+s-1} a_i x^i \\ y'(x) &= \sum_{i=0}^{r+s-1} i a_i x^{i-1} \end{aligned}$$

also,

$$y''(x) = \sum_{i=0}^{r+s-1} (i-1) a_i x^{i-2}$$

where $r + s - 1 = 7 + 2 - 1 = 8$

Hence,

$${}^k y(x) = \sum_{i=0}^8 a_i x^i$$

and

$${}^k y''(x) = \sum_{i=0}^8 (i-1) a_i x^{i-2} = f(x, y, y')h^2$$

Now let us consider the point of interpolation at $x_{n+\frac{1}{3}}$ and $x_{n+\frac{2}{3}}$. Also, points of collocation belong to the interval $[0, 2]$ i.e. $x_n, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}$ and x_{n+2} .

Gives the following system of non -linear equation of the form:

$$\begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} & \frac{1}{729} & \frac{1}{2187} & \frac{1}{6561} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} & \frac{64}{729} & \frac{128}{2187} & \frac{256}{6561} \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{2}{h^2} & \frac{4}{3h^2} & \frac{20}{27h^2} & \frac{10}{27h^2} & \frac{14}{81h^2} & \frac{56}{729h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{4}{h^2} & \frac{16}{3h^2} & \frac{160}{27h^2} & \frac{160}{27h^2} & \frac{448}{27h^2} & \frac{3584}{729h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{8}{h^2} & \frac{64}{3h^2} & \frac{1280}{27h^2} & \frac{2560}{27h^2} & \frac{14336}{81h^2} & \frac{229376}{729h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{10}{h^2} & \frac{100}{3h^2} & \frac{2500}{27h^2} & \frac{6250}{27h^2} & \frac{43750}{81h^2} & \frac{875000}{729h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} & \frac{1344}{h^2} & \frac{3548}{h^2} \end{pmatrix} \quad (8)$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \end{bmatrix}$$

Solving equation (8) gives the coefficients $a_i, i = 0, \dots, 8$. Those values are then substitutes into equation (6) to gives Continuous hybrid multistep method of the form.

$$\begin{aligned} y(x) &= \sum_{i=0}^k \alpha_i(x) y_{n+i} + \sum_{\bar{v}}^k \alpha_{\bar{v}} y + \\ &\left(\sum_{i=0}^k \beta_{i=0}(x) f_{n+1} + \sum_{\bar{v}}^k \beta_{\bar{v}} f_{n+\bar{v}} \right) \end{aligned} \quad (9)$$

where $k=2, \bar{v} = \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}$ yield the parameter α_i and $\beta_i, i = 0, \bar{v}, 1$ as

$$\begin{aligned} \alpha_{n+\frac{1}{3}} &= -3t + 2 \\ \alpha_{n+\frac{2}{3}} &= 3t - 1 \end{aligned}$$

where $t = \frac{x-x_n}{h}$

So we have

$$\beta_0 = h^2 \left[\frac{21}{32}t^6 - \frac{459}{4480}t - \frac{27}{160}t^7 + \frac{1}{2}t^2 + \frac{81}{4480}t^8 + \frac{863}{108894} - \frac{49}{40}t^3 - \frac{441}{320}t^5 + \frac{203}{120}t^4 \right]$$

$$\beta_{\frac{1}{3}} = h^2 \left[\frac{-243}{2240}t^8 + \frac{27}{28}t^7 - \frac{279}{80}t^6 + \frac{261}{40}t^5 - \frac{261}{40}t^4 + 3t^3 - \frac{27623}{60480}t + \frac{8999}{90720} \right]$$

$$\beta_{\frac{2}{3}} = h^2 \left[\frac{243}{896}t^8 - \frac{513}{224}t^7 + \frac{1233}{160}t^6 - \frac{4149}{320}t^5 + \frac{351}{32}t^4 - \frac{15}{4}t^3 + \frac{18689}{120960}t - \frac{769}{181440} \right]$$

$$\beta_1 = h^2 \left[\frac{-81}{224}t^8 + \frac{81}{28}t^7 - \frac{363}{40}t^6 + \frac{279}{20}t^5 - \frac{127}{12}t^4 + \frac{10}{3}t^3 - \frac{139}{864}t + \frac{1987}{136080} \right]$$

$$\beta_{\frac{4}{3}} = h^2 \left[\frac{243}{896}t^8 - \frac{459}{224}t^7 + \frac{963}{160}t^6 - \frac{2763}{320}t^5 + \frac{99}{16}t^4 - \frac{15}{8}t^3 + \frac{10921}{120960}t - \frac{1609}{181440} \right]$$

$$\beta_{\frac{5}{3}} = h^2 \left[\frac{-243}{2240}t^8 + \frac{27}{35}t^7 - \frac{171}{80}t^6 + \frac{117}{40}t^5 - \frac{81}{40}t^4 + \frac{3}{5}t^3 - \frac{347}{12096}t + \frac{263}{90720} \right]$$

$$\beta_2 = h^2 \left[\frac{81}{4480}t^8 - \frac{27}{224}t^7 + \frac{51}{160}t^6 - \frac{27}{64}t^5 + \frac{137}{480}t^4 - \frac{1}{12}t^3 + \frac{479}{120960}t - \frac{221}{544320} \right]$$

Evaluating the above continuous coefficients at $t = 0, 1, \frac{4}{3}, \frac{5}{3}, 2$ (non-interpolating points), we obtain

Evaluating the above continuous coefficients at $t = 0, 1, \frac{4}{3}, \frac{5}{3}, 2$ (non-interpolating points), we obtain

$$y(x) = \sum_{i=0}^k \alpha_i(x) y_{n+i} + \sum_{\bar{v}} \alpha_{\bar{v}} y + \left(\sum_{i=0}^k \beta_{i=0}(x) f_{n+1} + \sum_{\bar{v}} \beta_{\bar{v}} f_{n+\bar{v}} \right)$$

$$y_{n+1} = -y_{n+\frac{1}{3}} + 2y_{n+\frac{2}{3}} + h^2 \left(\frac{-221}{544320}f_n + \frac{977}{90720}f_{n+\frac{1}{3}} + \frac{16451}{181440}f_{n+\frac{2}{3}} + \frac{1357}{136080}f_{n+1} + \frac{71}{181440}f_{n+\frac{4}{3}} - \frac{31}{90720}f_{n+\frac{5}{3}} + \frac{31}{544320}f_{n+2} \right)$$

$$y_{n+\frac{4}{3}} = -2y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}} + h^2 \left(\frac{-137}{181440}f_n + \frac{209}{10080}f_{n+\frac{1}{3}} + \frac{433}{2240}f_{n+\frac{2}{3}} + \frac{4927}{45360}f_{n+1} + \frac{257}{20160}f_{n+\frac{4}{3}} - \frac{1}{672}f_{n+\frac{5}{3}} + \frac{31}{181440}f_{n+2} \right)$$

$$y_{n+\frac{5}{3}} = -3y_{n+\frac{1}{3}} + 4y_{n+\frac{2}{3}} + h^2 \left(\frac{-19}{181440}f_n + \frac{17}{560}f_{n+\frac{1}{3}} + \frac{2987}{10080}f_{n+\frac{2}{3}} + \frac{4927}{22680}f_{n+1} + \frac{389}{3360}f_{n+\frac{4}{3}} + \frac{41}{5040}f_{n+\frac{5}{3}} - \frac{11}{90720}f_{n+2} \right)$$

$$y_{n+2} = -4y_{n+\frac{1}{3}} + 5y_{n+\frac{2}{3}} + h^2 \left(\frac{-95}{54432}f_n + \frac{389}{9072}f_{n+\frac{1}{3}} + \frac{7085}{18144}f_{n+\frac{2}{3}} + \frac{4633}{13608}f_{n+1} - \frac{3893}{18144}f_{n+\frac{4}{3}} + \frac{1061}{9072}f_{n+\frac{5}{3}} + \frac{409}{54432}f_{n+2} \right)$$

Differentiating (11) yields

$$y'(x) = \frac{1}{h} \sum_{i=0}^k \alpha_i^1(x) y_{n+i} + \frac{1}{h} \sum_{\bar{v}} \alpha_{\bar{v}}^1 y_{n+\bar{v}} + h \left(\sum_{i=0}^k \beta_i^1(x) f_{n+1} + \sum_{\bar{v}} \beta_{\bar{v}}^1 f_{n+\bar{v}} \right)$$

where $\alpha_{\frac{1}{3}}^1(t) = \frac{-3}{h}, \alpha_{\frac{2}{3}}^1(t) = \frac{3}{h}$

$$\beta_0^1(t) = h \left[\frac{81}{560}t^7 - \frac{189}{160}t^6 + \frac{63}{16}t^5 - \frac{441}{40}t^4 + \frac{203}{30}t^3 + \frac{147}{40}t^2 + t - \frac{459}{4480} \right]$$

$$\beta_{\frac{1}{3}}^1(t) = h \left[\frac{-243}{280}t^7 + \frac{27}{4}t^6 - \frac{837}{40}t^5 + \frac{261}{8}t^4 - \frac{261}{10}t^3 + 9t^2 - \frac{27623}{60480} \right]$$

$$\beta_{\frac{2}{3}}^1(t) = h \left[\frac{243}{280}t^7 - \frac{513}{32}t^6 + \frac{3699}{80}t^5 - \frac{4149}{64}t^4 + \frac{351}{8}t^3 + \frac{45}{4}t^2 + \frac{18689}{120960} \right]$$

$$\beta_1^1(t) = h \left[\frac{-81}{28}t^7 + \frac{81}{4}t^6 - \frac{1089}{20}t^5 + \frac{279}{4}t^4 - \frac{127}{3}t^3 + 10t^2 - \frac{139}{864} \right]$$

$$\beta_{\frac{4}{3}}^1(t) = h \left[\frac{243}{112}t^7 - \frac{459}{32}t^6 + \frac{2889}{80}t^5 - \frac{2763}{64}t^4 + \frac{99}{4}t^3 + \frac{45}{8}t^2 + \frac{10921}{120960} \right]$$

$$\beta_{\frac{5}{3}}^1(t) = h \left[\frac{-243}{280}t^7 + \frac{27}{5}t^6 - \frac{513}{40}t^5 + \frac{117}{8}t^4 - \frac{81}{10}t^3 + \frac{9}{5}t^2 - \frac{347}{12096} \right]$$

$$\beta_2^1(t) = h \left[\frac{81}{560}t^7 - \frac{27}{32}t^6 + \frac{153}{80}t^5 - \frac{135}{64}t^4 + \frac{137}{120}t^3 + \frac{1}{4}t^2 + \frac{479}{120960} \right]$$

Evaluating equation (16) above at $t = 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$

$$hy_{n+1}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-459}{4480}f_n - \frac{27623}{60480}f_{n+\frac{1}{3}} + \frac{18689}{120960}f_{n+\frac{2}{3}} - \frac{139}{864}f_{n+1} + \frac{10921}{120960}f_{n+\frac{4}{3}} - \frac{347}{12096}f_{n+\frac{5}{3}} + \frac{479}{120960}f_{n+2} \right]$$

$$hy_{n+\frac{4}{3}}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{199}{72576}f_n - \frac{1973}{20160}f_{n+\frac{1}{3}} - \frac{18}{128}f_{n+\frac{2}{3}} + \frac{4157}{90720}f_{n+1} - \frac{851}{40320}f_{n+\frac{4}{3}} + \frac{41}{6720}f_{n+\frac{5}{3}} + \frac{289}{120960}f_{n+2} \right]$$

$$y_{n+\frac{2}{3}}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-731}{362880}f_n - \frac{13}{320}f_{n+\frac{1}{3}} + \frac{6347}{40320}f_{n+\frac{2}{3}} - \frac{3971}{90720}f_{n+1} + \frac{257}{13440}f_{n+\frac{4}{3}} - \frac{109}{20160}f_{n+\frac{5}{3}} + \frac{253}{262880}f_{n+2} \right]$$

$$hy_{n+1}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-1}{1920}f_n + \frac{1537}{60480}f_{n+\frac{1}{3}} + \frac{39587}{120960}f_{n+\frac{2}{3}} + \frac{4927}{30240}f_{n+1} - \frac{2201}{120960}f_{n+\frac{4}{3}} + \frac{209}{6720}f_{n+\frac{5}{3}} - \frac{43}{120960}f_{n+2} \right]$$

$$hy_{n+\frac{4}{3}}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-571}{362880}f_n + \frac{691}{20160}f_{n+\frac{1}{3}} + \frac{1299}{4480}f_{n+\frac{2}{3}} + \frac{33533}{90720}f_{n+1} + \frac{1223}{8064}f_{n+\frac{4}{3}} + \frac{79}{6720}f_{n+\frac{5}{3}} + \frac{59}{51840}f_{n+2} \right]$$

$$hy_{n+\frac{5}{3}}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-29}{362880}f_n + \frac{51}{2240}f_{n+\frac{1}{3}} + \frac{13313}{40320}f_{n+\frac{2}{3}} + \frac{5081}{18144}f_{n+1} - \frac{5519}{13440}f_{n+\frac{4}{3}} + \frac{73}{576}f_{n+\frac{5}{3}} + \frac{1313}{362880}f_{n+2} \right]$$

$$hy_{n+2}^1 + 3y_{n+\frac{1}{3}}^1 - 3y_{n+\frac{2}{3}}^1 = h^2 \left[\frac{-13}{2688}f_n + \frac{3481}{60480}f_{n+\frac{1}{3}} + \frac{5293}{24193}f_{n+\frac{2}{3}} + \frac{14719}{30240}f_{n+1} + \frac{2671}{17280}f_{n+\frac{4}{3}} + \frac{29369}{576}f_{n+\frac{5}{3}} + \frac{12287}{120960}f_{n+2} \right]$$

Equation (9) is evaluated at non-interpolating points, i.e, x_n , $x_{n+\frac{1}{3}}$, $x_{n+\frac{2}{3}}$, x_{n+1} , $x_{n+\frac{4}{3}}$, $x_{n+\frac{5}{3}}$ and x_{n+2} , while Equation (16) is evaluated at all points, and this yields the following equation in matrix form:

$$A\gamma_m = BR_1 + CR_2 + DR_3 \quad (19)$$

Where

$$A^* = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{3}{h} & \frac{-3}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (20)$$

$$\gamma_m = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+\frac{5}{3}} \\ y_{n+2} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+\frac{5}{3}} \\ y_{n+2} \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, R_1 = \begin{bmatrix} y_n \\ y_n \end{bmatrix}$$

$$C = \begin{pmatrix} 863h^2 \\ 108864 \\ -221h^2 \\ 544320 \\ -137h^2 \\ 181440 \\ -19h^2 \\ 1881440 \\ -95h^2 \\ 54432 \\ -459h \\ 4480 \\ 199h \\ 72576 \\ -731h \\ 362880 \\ -h \\ 1920 \\ -571h \\ 362880 \\ -29h \\ 362880 \\ -13h \\ 2688 \end{pmatrix}, R_2 = [f_n], R_3 = \begin{bmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \end{bmatrix} \quad (21)$$

$$D = \begin{bmatrix} 8999h^2 & -769h^2 & 1987h^2 & -1609h^2 & 263h^2 & -221h^2 \\ 90720 & 181440 & 136080 & 181440 & 90720 & 544320 \\ 977h^2 & 16451h^2 & 1357h^2 & 71h^2 & -31h^2 & 31h^2 \\ 90720 & 181440 & 136080 & 181440 & 90720 & 544320 \\ 209h^2 & 433h^2 & 4927h^2 & 257h^2 & -h^2 & 31h^2 \\ 10080 & 2240 & 45360 & 20160 & 672 & 181440 \\ 17h^2 & 2987h^2 & 4927h^2 & 389h^2 & 41h^2 & -11h^2 \\ 560 & 10080 & 22680 & 3360 & 5040 & 90720 \\ 389h^2 & 7085h^2 & 4633h^2 & 3893h^2 & 1061h^2 & 409h^2 \\ 9072 & 18144 & 13698 & 18144 & 9072 & 54432 \\ -27623h & 18689h & -139h & 10921h & 347h & 479h \\ 60480 & 120960 & 864 & 120960 & 12096 & 120960 \\ -1973h & -18h & 4157h & -851h & 41h & 289h \\ 20160 & 128 & 90720 & 40320 & 6720 & 362880 \\ 13h & 6347h & -3971h & 257h & 109h & 253h \\ 320 & 40320 & 90720h & 13440 & 20160 & 262880 \\ 1537h & 39587h & 4927h & -2201h & 209h & -43h \\ 60480 & 120960 & 30240 & 120960 & 60480 & 120960 \\ 691h & 1299h & 33533h & 1223h & -79h & 59h \\ 20140 & 4480 & 90720 & 8064 & 6720 & 51840 \\ 51h & 13313h & 5081h & 5519h & 73h & -1313h \\ 2240 & 40320 & 18144 & 13440 & 576 & 362880 \\ 3481h & 5295h & 14719h & 2671h & 29369h & 12287h \\ 60480 & 24192 & 30240 & 17280 & 60480 & 120960 \end{bmatrix}$$

Multiplying equation (19) by A^{-1} gives the hybrid Block method as shown below

$$I\gamma_m = B^*R_1 + C^*R_2 + D^*R_3 \quad (22)$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B^* = \begin{bmatrix} 1 & -\frac{h}{3} \\ 1 & -\frac{2h}{3} \\ 1 & -h \\ 1 & -\frac{4}{3} \\ 1 & -\frac{5h}{3} \\ 1 & -2h \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}, \quad C^* = \begin{bmatrix} \frac{28549h^2}{1088640} \\ \frac{1027h^2}{17010} \\ \frac{253h^2}{2688} \\ \frac{1088h^2}{8505} \\ \frac{17715h^2}{1088640} \\ \frac{41h^2}{210} \\ \frac{19087h}{181440} \\ \frac{1139h}{11340} \\ \frac{137h}{1344} \\ \frac{286h}{2835} \\ \frac{3715h}{36288} \\ \frac{41h}{420} \end{bmatrix}, \quad (23)$$

$$D^* = \begin{bmatrix} \frac{275h^2}{5184} & -\frac{5717h^2}{120960} & \frac{19621h^2}{272160} & -\frac{7703h^2}{362880} & -\frac{3991h^2}{120960} & -\frac{199h^2}{217728} \\ \frac{194h^2}{945} & -\frac{8h^2}{81} & \frac{788h^2}{8505} & -\frac{97h^2}{1890} & -\frac{11447h^2}{181440} & -\frac{19h^2}{8505} \\ \frac{165h^2}{448} & -\frac{267h^2}{4489} & \frac{5h^2}{32} & -\frac{363h^2}{4480} & -\frac{419h^2}{4489} & -\frac{47h^2}{13440} \\ \frac{1504h^2}{2835} & -\frac{8h^2}{945} & \frac{2624h^2}{8505} & -\frac{8h^2}{81} & -\frac{539h^2}{4320} & -\frac{8h^2}{1701} \\ \frac{8375h^2}{12096} & \frac{3125h^2}{72576} & \frac{25625h^2}{54432} & -\frac{625h^2}{24192} & -\frac{10541h^2}{72576} & -\frac{1375h^2}{217728} \\ \frac{6h^2}{7} & \frac{3h^2}{35} & \frac{68h^2}{105} & \frac{3h^2}{70} & -\frac{149h^2}{2240} & 0 \\ \frac{166771h}{30240} & -\frac{35699h}{120960} & \frac{586h}{2835} & -\frac{523h}{7560} & -\frac{10183h}{120960} & -\frac{41h}{12960} \\ \frac{94h}{189} & \frac{11h}{3780} & \frac{332h}{2835} & -\frac{269h}{3780} & -\frac{10267h}{120560} & -\frac{595729h}{198737280} \\ \frac{27h}{56} & \frac{387h}{2240} & \frac{34h}{105} & -\frac{243h}{2240} & -\frac{10921h}{120960} & -\frac{29h}{6720} \\ \frac{464h}{945} & \frac{128h}{945} & \frac{1504h}{2835} & \frac{58h}{945} & -\frac{12343}{120960} & -\frac{8h}{2835} \\ \frac{725h}{1512} & \frac{2125h}{12096} & \frac{250h}{567} & \frac{3875h}{12096} & \frac{4409h}{120960} & -\frac{275h}{36288} \\ \frac{18h}{35} & -\frac{18689h}{18689} + \frac{5293h^2}{24192} & \frac{139h}{864} + \frac{14719h^2}{30240} & \frac{9h}{140} & \frac{253h}{640} & \frac{41h}{420} \end{bmatrix}$$

The above blocked method is of uniform order $P = (7,7,7,7,7,7)^T$ with an error constant c_{p+2}

Analysis of the method

Zero stability

The two step hybrid block method (Jator, 2013) is said to be zero stable if no root of the first characteristic equation $\rho(R)$ has modulus greater than one ,i.e $|R_m| \leq 1$ and if $R_m = 1$, then the

multiplicity of R_m must not exceed 2.

The characteristic function of the new derived method is given as below;

$$\rho(R) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{h}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{2h}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{4h}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{5h}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 0 \quad (24)$$

$$\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{h}{3} \\ 0 & \lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{2h}{3} \\ 0 & 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -h \\ 0 & 0 & 0 & \lambda & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{4h}{3} \\ 0 & 0 & 0 & 0 & \lambda & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{5h}{3} \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 & 0 & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix} = 0$$

$$\text{Thus, } \lambda^{10}(\lambda - 1)^2 = 0 \tag{25}$$

$$\text{Where } \lambda_i = \begin{cases} 0 & \text{if } i = 1(1)10 \\ 1 & \text{if } i = 11,12. \end{cases} \tag{.26}$$

Hence, the developed method is zero stable.

Order of the method.

According to Yusuph & Onumanyi (2005) the order of the new method in equation Leentvaar (1973) is obtained by using the Taylor series and it is found that the developed scheme has an order of $(7,7,7,7,7,7,7)^T$ with an error constant vector of; $[1.732063 \times 10^{-7}, 4.1014112 \times 10^{-7}, 6.582812 \times 10^{-7}, -1.745322 \times 10^{-6}, 1.531063 \times 10^{-7}, 1.732063 \times 10^{-6}, -1.666063 \times 10^{-5}]$

Consistency

Hybrid block method is said to be consistent if it has an order that is more than or equal to one. The order is order two. Therefore, the method is consistent.

Convergence

Zero stability and consistency are sufficient conditions for a linear multistep method to be convergent (Onumanyi, 1981). Since the new hybrid block method is zero stable and consistent, it can be concluded that the method is convergent.

Implementation of the Method

The initial starting value at each block is obtained by using Taylor series method. Then, the outcomes are calculated and corrected using the new scheme. For the next block, the same technique are repeated to compute the approximate values of $x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}, x_{n+1}, x_{n+\frac{4}{3}}$ and $x_{n+\frac{5}{3}}$ until the end of the integrated intervals. During the calculations of the iteration, the final value of y_{n+1} are taken as the initial values for the next iteration.

Numerical Experiment/Result

In this section, the performance of the developed two step hybrid block scheme is examined. The tables below shows the numerical results of the new developed scheme with exact solution for solving the problem and the result of the developed scheme are more accurate than that of Dura & Osneagu (2010) which was executed by six step method for solving the equation

Problem: Consider for purposes solving for the value of call option with strike price $k=100$. The risk – free interest rate $r = 0.12$, the time to expiration is $T = 1$ measured in years, and the volatility is $= 0.10$. The value of the call option can have a range of $70 \leq s \leq 130$.

Table 1. Comparison between the explicit method and Hybrid method

| Stock Price | Explicit method | Hybrid method | Exact method |
|-------------|-----------------|---------------|--------------|
| 4 | 1.631276e-06 | 1.168607e-06 | 1.067322e-06 |
| 8 | 0.146176 | 0.149236 | 0.149335 |
| 10 | 0.906372 | 0.916099 | 0.916291 |
| 16 | 6.236436 | 6.252284 | 6.252287 |
| 20 | 10.223892 | 10.246903 | 10.247014 |

Table 2. Approximate value by hybrid method and the exact value

| Stock Price | Hybrid method | Exact method |
|-------------|---------------|--------------|
| 4 | 1.168607e-06 | 1.067322e-06 |
| 8 | 0.149236 | 0.149335 |
| 10 | 0.916099 | 0.916291 |
| 16 | 6.252286 | 6.252287 |
| 20 | 10.246904 | 10.247014 |

Comparison with Other Numerical Scheme

In this part, a comparison is made between our Hybrid Scheme with another scheme that was used in finding the approximate solution of Black- Scholes model which is explicit method For a European call option with $0 \leq S \leq 20, T = 0.25, k = 10, r = 0.1, \sigma = 0.4$, with temporal grid size of $N = 2000$ and spatial grid size $M = 200$, the explicit method and Hybrid method were used to set the table above.

From Table 1, it is seen that hybrid scheme gives better results than explicit method. But the results obtained by the scheme that was developed here are not much close to the exact value. But the good news is that if the temporal grid points are increased up to $N = 41000$, spatial grid points up to $M = 1000$ and set the other parameters value as above, then the following value (Table 2) were found much more better for different stock prices.

The numerical result confirm that the proposed scheme produces a better accuracy if compared with the existing methods

Conclusions

In this article, a two-step block method with five off-step points is derived via the interpolation collocation approach. The developed method is consistent, Zero Stable, Convergent and of order seven. The relative error of the hybrid scheme was estimated by comparing the numerical solution with the analytical solution in Li-norms. The numerical simulation results are seen in good agreement with well-known qualitative behavior of the Black Scholes PDE. Also, a comparison is presented between the hybrid method and the result obtained by another work using the explicit method where our result is found more accurate than that of the explicit method.

Conflicts of interest: The authors declare no conflict of interest.

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