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NUMERICAL SOLUTION OF THIRD ORDER THREE-POINT BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS USING VARIATIONAL-COMPOSITE HYBRID FIXED POINT ITERATIVE METHOD

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ABSTRACT
This paper explores variational–composite hybrid fixed point iterative scheme for the solution of third order three-point boundary value problems. The method shows a strong convergence which makes it an efficient and reliable technique for finding approximate analytical solutions for third order three-point boundary value problems of ordinary differential equations with variable coefficients. From the numerical experiments carried out, the accuracy of the method was confirmed through the order of convergence obtained.

Keywords: Variational, Iteration, Composite, Hybrid, Fixed point, Order.

INTRODUCTION

With the rapid development of linear and nonlinear science, many numerical methods have been developed and used in an attempt to solve various types of boundary value problems (BVPs), such as fixed point iteration method, homotopy perturbation method (HPM) and variational iteration method (VIM). The existence and multiplicity of solutions of three-point boundary value problems have been extensively studied in the literature. Multi-point boundary value problems arise in various areas of applied mathematics and engineering. In Al-Mustapha & Adeboye (2017), the concept, development and applications of variational-composite hybrid fixed iteration method was discussed. The method shows a strong convergence which makes it an efficient and reliable technique for finding approximate analytical solutions for both linear and non-linear three point boundary value problems. Adeboye (1999) first defined a H^1 Galerkin method on boundary value problems and then converted it to a fixed point iterative process which led to super convergence. Li & Wu (2011)a and Li & Wu (2011)b introduced the reproducing kernel method for solving linear singular fourth order four-point boundary value problems. Also, they presented the analytical approximation of nonlinear singular fourth order fourth-point boundary value problems by combining homotopy perturbation method and reproducing kernel method for solving linear singular fourth order four-point boundary value problems. Geng & Cui (2009) presented a method for solving nonlinear multi-point boundary value problems by combining homotopy perturbation and variational iteration methods. John *et al*. (2003) presented some results on existence and nonexistence of positive solutions of fourth order nonlinear differential equation three-point boundary value problems. Geng (2009a) and Geng (2009b) introduced a new reproducing kernel Hilbert space method for solving nonlinear fourth order boundary value problems. Iyase (2010) presented some results concerning the existence of solutions for the fourth order three-point boundary value problems. Tatari & Dehgan (2006) used the Adomian decomposition method for solving multi-point boundary value problems.

In this paper, we aim to apply variational-composite hybrid fixed point iterative method proposed by Al-Mustapha and Adeboye (2017) to differential equations of the third order three-point boundary value problems with variable coefficients.

MATERIALS AND METHODS

Variational Iteration Method (VIM)

This method is implemented to give approximate and analytical solutions for a class of boundary value problems and it produces the solutions in terms of convergent series.

General Lagrange Multipliers are introduced to construct correction functional for the systems in this method. The Multipliers in the functional can be identified by the variational theory. The starting function can be freely chosen with possible unknown constraints which can be determined by imposing the boundary/initial conditions.

To illustrate the basic concept of the technique, consider the following nonlinear system

$$Lw + Nw = g(x) \qquad (1)$$

Where L and $N$are linear and nonlinear operators respectively and $g(x)$ is a forcing term. The correction functional for the problem (1) can be approximately constructed as follows:

$$w_{m+1}(x) = w_m(x) + \int_0^x \lambda(Lw_m(\xi) + N\widetilde{w_m}(\xi) - g(\xi)d\xi \ , \quad m \geq 0 \qquad (2)$$

where, λ is a general Lagrange Multiplier, that can be identified optimally via variational theory; here, w_m is the mth approximate solution and $\widetilde{w_m}$ is considered as restricted variation, i.e., $\delta\widetilde{w_m} = 0$, making the correction functional (2) stationary

Variational-Composite hybrid fixed point iterative method (VCHFPIM)

Let A be Hilbert space, $T: A \rightarrow A$ a nonexpensive mapping with $F(T) \neq \emptyset$ and $j(resp.k): A \rightarrow A$ and $\tau_j(resp.\tau_k)$ be *strongly* *monotone* and $L_j(resp. L_k)$ be Lipschitzian mappings. For any $x_1 \in A$, $\{\alpha_n\}$ is generated by

$$\left.\begin{array}{l} y_{n+1} = \alpha_n y_n + (1-\alpha_n)T_j x_n \\ x_n = \beta_n y_n + (1-\beta_n)T_k y_n \end{array}\right\}, \qquad n \geq 1 \tag{3}$$

which is a Hybrid Iteration process, where T_j and T_k are the given operators and $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\}_{n\subset\mathbb{N}}$ $(0,1)$ are sequences satisfying appropriate conditions. From equation (3) we have

$$y_{n+1} = \alpha_n y_n + (1-\alpha_n)T_j[\beta_n y_n + (1-\beta)T_k y_n \tag{4}$$

Suppose we set

$$T_n = T_j[\beta_n y_n + (1-\beta)T_k \tag{5}$$

Then

$$y_{n+1} = \alpha_n y_n + (1-\alpha_n)T y_n \tag{6}$$

Next, we consider the differential equation of the form

$$y''' + q(x)y'' + r(x)y' + s(x)y = f(x) \tag{7}$$

with the boundary conditions

$$y(a) = \beta_1, y'(\xi) = \beta_2, y''(b) = \beta_3\,, \qquad a < \xi < b \tag{8}$$

where $q, r, s, f \in [a,b]$ and $\beta_1, \beta_2, \beta_3$ are constants.

Let

$$y''' = f(x) - (q(x)y'' + r(x)y' + s(x)y \tag{9}$$

Then, using (6) the fixed point iterative scheme is obtained as follows:

$$\begin{aligned} y'''_{n+1} = \alpha_n y'''_n + (1-\alpha_n)T y'''_n &= \alpha_n y'''_n \\ &\quad + (1-\alpha_n)[f(x) - q(x)y''_n - r(x)y'_n \\ &\quad - s(x)] \end{aligned} \tag{10}$$

converges for $0 \leq \alpha_n \leq 1$. This is the proposed iterative method for the solution of boundary value problems.

Numerical Applications

Problem 1:

Consider third order linear differential equation with variable coefficients

$$x^3y''' + 2x^2y'' = -6\ln x\,, \qquad x \in [1,e] \tag{11}$$

subject to boundary conditions

$$y(1) = 0, y''\left(e^{\frac{1}{2}}\right) = 9, y(e) = 4 \tag{12}$$

$$y_E(x) = \ln^3 x + 3\ln^2 x \tag{13}$$

Equation (11) is transformed as follows:

Assume from (11)

$$x = e^z, z = \ln x\,, z^2 = \ln^2 x \tag{14}$$

$$x = e^z, z = \ln x, z^2 = \ln^2 x$$

Then, (11), (12) and (13) become

$$(y'''_{z+1} - 3y''_z + 2y'_z) + 2y''_z - 2y'_z = y'''_z - y''_z = -6z \tag{15}$$

and

$$y_z(0) = 0, y''_z\left(\frac{1}{2}\right) = 9, y_z(1) = 4 \tag{16}$$

$$y_E(z) = z^3 + 3z^2 \tag{17}$$

Applying variational iteration method, we construct a correction functional as

$$\begin{aligned} y_{n+1}(z) = y_n(z) - \int_0^z \frac{1}{2}(t-z)^2(y'''_n(t) - y''_n(t) \\ + 6t)dt, \qquad n = 0,1.\cdots \end{aligned} \tag{18}$$

where,

$$\lambda(t) = -\frac{1}{2!}(t-z)^2 \tag{19}$$

Starting with the initial approximation, $y_0(z) = az + \frac{1}{2}bz^2$ in equation (18).

Problem 2:

Consider third order linear differential equation with variable coefficients

$$x^3y''' + 3x^2y'' = \ln^2 x\,, \quad x \in [1,e] \tag{20}$$

subject to boundary conditions

$$y(1) = 0, y''\left(e^{\frac{1}{2}}\right) = -1, y(e) = \frac{2}{3} \tag{21}$$

$$y_E(x) = \frac{3}{2} - \frac{3x}{2(1-e)} + \frac{3e}{2(1-e)x} - \frac{\ln^3 x}{3} - 2\ln x \tag{22}$$

RESULTS

The results of problem 1 are presented in Tables I, II and III, while those of Problem 2 are shown in Tables III, IV and V. All computations were carried out using MAPLE 2017 software package.

Table I: Comparison of results of Problem 1 when $n = 5, \alpha = 1/2$

z	Exact	VIM	VCHFPIM	Error(VIM)	Error(VCHPIM)
0.0	0.000	0.000000000	0.000000000	0.000000000	0
0.1	0.031	0.025252116	0.031033714	5.747883331E-3	3.371429100E-5
0.2	0.128	0.122034133	0.128064848	5.965866710E-3	6.484841000E-5
0.3	0.297	0.296922800	0.297095714	7.720001000E-5	9.571431000E-5
0.4	0.544	0.556541866	0.544128330	1.254186671E-2	1.283305100E-4
0.5	0.875	0.907552083	0.875163217	3.255208301E-2	1.632177100E-4
0.6	1.296	1.356635200	1.296197639	6.063520010E-2	1.976391000E-4
0.7	1.813	1.910471967	1.813223151	9.747196710E-2	2.231511000E-4
0.8	2.432	2.575714133	2.432222322	1.437141331E-1	2.223221000E-4
0.9	3.159	3.358950450	3.159164436	1.999504501E-1	1.644361000E-5
1.0	4.000	4.266666667	4.000000000	2.666666671E-1	0.000000000

Table II: Comparison of results of Problem 2 when $n = 5, \alpha = 1$

z	Exact	VIM	VCHFPIM	Error(VIM)	Error (VCHPIM)
0.0	0.000	0.000000000	0.000000000	0.000000000	0.0
0.1	0.031	0.025252116	0.030999999	5.374788333E-3	1.1E-11
0.2	0.128	0.122034133	0.128000000	0.005965866E-3	0.0
0.3	0.297	0.296922800	0.296999999	7.720001000E-5	1.1E-10
0.4	0.544	0.556541866	0.544000000	1.254186671E-2	0.0
0.5	0.875	0.907552083	0.875000000	3.255208301E-2	0.0
0.6	1.296	1.356635200	1.296000000	6.063520010E-2	0.0
0.7	1.813	1.910471967	1.813000000	9.747196710E-2	0.0
0.8	2.432	2.575714133	2.432000001	1.437141331E-1	1.10E-9
0.9	3.159	3.358950450	3.159000001	1.999504501E-1	1.10E-9
1.0	4.000	4.266666667	4.000000000	2.66666667E-1	0.0

Table III: Error estimates of Problem 1 for different derivatives when $n = 5$

z	Exact (1st derivative)	Exact (2nd derivative)	Error(VCHPIM) (1st derivative)	Error(VCHPIM) (2nd derivative)
0.0	0.00	6.0	5.3E-10	1.2E-8
0.1	0.63	6.6	1.1E-10	2.10E-9
0.2	1.32	7.2	1.1E-9	0.0
0.3	2.07	7.8	0.0	0.0
0.4	2.88	8.4	0.0	0.0
0.5	3.75	9.0	0.0	0.0
0.6	4.68	9.6	1.1E-9	0.0
0.7	5.67	10.2	0.0	0.0
0.8	6.72	10.8	1.1E-9	0.0
0.9	7.83	11.4	0.0	0.0
1.0	9.00	12.0	1.1E-9	2.1E-8

Table IV: Comparison of results of Problem 2 when $n = 4,\ \alpha = 1/2$

z	Exact	VIM	VCHFPIM	Error(VIM)	Error(VCHPIM)
0.0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.117294685	0.117174718	0.116637758	1.199674100E-4	6.569275100E-4
0.2	0.220755793	0.220728171	0.219881078	2.762281000E-5	8.747156100E-4
0.3	0.311443822	0.311721775	0.310652206	2.779534100E-4	7.916155100E-4
0.4	0.390331456	0.391131280	0.389808558	7.998246100E-4	5.228980100E-4
0.5	0.458333333	0.459876545	0.458166485	1.543212410E-3	1.668476000E-4
0.6	0.516335211	0.518850766	0.516524934	2.515555610E-3	1.897233100E-4
0.7	0.565222845	0.568949431	0.565689131	3.726586510E-3	4.662865100E-4
0.8	0.605910871	0.611099297	0.606494460	4.037681281E-2	5.835899100E-4
0.9	0.639371980	0.646287683	0.639830686	6.915703810E-3	4.587064100E-4
1.0	0.666666666	0.675592373	0.666666666	8.925707610E-3	0.000000000

Table V: Comparison of results of Problem 2 when $n = 4,\ \alpha = 1$

z	Exact	VIM	VCHFPIM	Error(VIM)	Error(VCHPIM)
0.0	0.000000000	0.000000000	0	0.000000000	0.0000
0.1	0.117294685	0.117174718	0.117294735	1.199674100E-4	4.9110E-8
0.2	0.220755793	0.220728171	0.220755874	2.762281000E-5	8.1010E-8
0.3	0.311443822	0.311721775	0.311443901	2.779534100E-4	0.40000
0.4	0.390331456	0.391131280	0.390331504	7.998246100E-4	4.8310E-8
0.5	0.458333333	0.459876545	0.458333333	1.543212410E-3	0.0000
0.6	0.516335211	0.518850766	0.516335161	2.515555610E-3	4.9710E-8
0.7	0.565222845	0.568949431	0.565222764	3.726586510E-3	8.0410E-8
0.8	0.605910871	0.611099297	0.605910791	4.037681281E-2	7.9810E-8
0.9	0.639371980	0.646287683	0.639371930	6.915703810E-3	4.9210E-8
1.0	0.666666666	0.675592373	0.666666666	8.925707610E-3	0.0000

Table VI: Error estimates of Problem 2 for different derivatives when $n = 4$

z	Exact (1st derivative)	Exact (2nd derivative)	Error(VCHPIM) (1st derivative)	Error(VCHPIM) (2nd derivative)
0.0	1.245930120	- 1.500000000	5.27100E-7	9.71000E-8
0.1	1.101923175	- 1.382371981	4.29100E-7	1.96710E-6
0.2	0.969061404	- 1.276577539	1.61110E-7	3.17710E-6
0.3	0.846315330	- 1.179556178	1.66210E-7	3.14810E-6
0.4	0.732956886	- 1.088335211	4.28210E-7	1.93210E-6
0.5	0.628552129	- 0.999999999	5.29610E-7	0.00000
0.6	0.532956886	- 0.911664787	4.28210E-7	1.93031E-6
0.7	0.446315331	- 0.820443823	1.67110E-7	3.14971E-6
0.8	0.369061403	- 0.723422460	1.62710E-7	3.17931E-8
0.9	0.301923176	- 0.617628018	4.29010E-7	1.99821E-6
1.0	0.245930122	- 0.500000001	5.35810E-7	9.78100E-8

DISCUSSION

Interpretation

Tables I, II and III show the results of comparison of solutions of problem 1 by the new method with the variational iteration method and as well as the exact solution. It is clearly seen that the new method's approximate solutions are closer to the exact solution than the VIM. It is to be noted that even at the fifth iteration, the error is zero, that is, exact!

The comparison of the maximum absolute errors of the new method with the variational iteration method and the exact solution of problem 2 is given in Tables IV, V and VI. Table VI shows solution by the new method to the derivatives which shows that the new method is accurate, even for the derivatives, which is not common. From these tables, it can be observed that the new method gives better approximation than the VIM.

Conclusion

From the numerical experiments carried out and summary of tables shown the method shows a strong convergence which makes it an efficient and reliable technique for finding approximate analytical solutions for boundary value problems of third order ordinary differential equations with variable coefficients. The method is more accurate, less complicated and converges faster than many well-known methods in use.

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