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THE CONCEPT OF α-CUTS IN MULTI Q-FUZZY SET

A. I. Isah

Department of Mathematical Sciences, Kaduna State University, Kaduna State - Nigeria

Author's Email Address: ahmed.isah@kasu.edu.ng

ABSTRACT
The purpose of this paper is to introduce the concept of α-Cuts and their properties in multi Q-fuzzy sets. In addition, both first and second decomposition theorems were established and proved. It is shown that any Multi Q-fuzzy Set can be represented as the union of its special α-cuts as well as its special strong α-cuts.

Keywords: Fuzzy multiset, multi Q-fuzzy set, α-Cut.

1. INTRODUCTION

A fuzzy set which is a generalized set of objects occurring with a continuum of degrees of membership was introduced by (Zadeh, 1965); he further showed the application of α-cuts to fuzzy sets. For the basics of fuzzy set and its applications refer to (Brown, 1971; Singh *et al*., 2015; Goguen, 1967; Wygralak, 1989; Chutia et al., 2010; Dutta *et al*., 2011; Klir and Yuan, 1995; Kreinovich, 2013). Singh *et al*. (2014) studied α-cuts and some of its properties in fuzzy multisets. Multi Q-fuzzy set was studied in various contexts in (Adam and Hassan, 2014a; Adam and Hassan, 2014b; Adam and Hassan, 2015; Adam and Hassan, 2016); were its relevant applications were shown.
In this paper, α-cuts and their properties in Multi Q-fuzzy sets were studied.

2. Preliminaries

Definition 2.1 Multiset
An mset A drawn from the set X is represented by a function *Count* A or C_A defined as $C_A: X \longrightarrow \mathbb{N}$. One way of representing a multiset A from X with x_1 appearing k_1 times, x_2 appearing k_2 times etc., is $A = \{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$, where $x_i \in X$.

Let A and B be two msets drawn from a set X. Then
$A \subseteq B$ iff $C_A(x) \leq C_B(x)$ for all $x \in X$.
$A = B$ iff $C_A(x) = C_B(x)$ for all $x \in X$.
$A \cup B = max\{C_A(x), C_B(x)\}$ for all $x \in X$.
$A \cap B = min\{C_A(x), C_B(x)\}$ for all $x \in X$ (Jena *et al.*, 2001)

Definition 2.2 Fuzzy Multiset
A fuzzy multiset A is a multiset of pairs, where the first part of each pair is an element of a universe set X and the second part is the degree to which the first part belongs to that fuzzy multiset.
That is, $A: X \times I \longrightarrow \mathbb{N}$; where $I = [0,1]$ and $\mathbb{N}$ is the set of positive integers including 0 (Syropoulos, 2012).

Let A and B be fuzzy multisets. Then, Lengths $L(x;\ A)$ and $L(x;\ A, B)$ are respectively defined as

$$L(x;\ A) = \max\{j: \mu_A^j(x) \neq 0\};\ \text{and}$$
$$L(x;\ A, B) = \max\{L(x;\ A), L(x;\ B)\}.$$

For brevity, $L(x)$ for $L(x;\ A)$ or $L(x; A, B)$ is also used if no confusion arises.
Note that for defining an operation between two fuzzy multisets A and B, the lengths of the membership sequences $\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x)$ and $\mu_B^1(x), \mu_B^2(x), \ldots, \mu_B^p(x)$ need to be set equal.

Let A, B be fuzzy multisets. Then

$$A \cup B = \mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x), j = 1, \ldots, L(x), \forall x \in X.$$
$$A \cap B = \mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x), j = 1, \ldots, L(x), \forall x \in X.$$
$$A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x), j = 1, \ldots, L(x), \forall x \in X.$$

Thus, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Definition 2.3 Multi Q-fuzzy Set
Let I be a unit interval $[0, 1]$, k be a positive integer, U be a universal set and Q be a non-empty set. A multi Q-fuzzy set A_Q in U and Q is a set of ordered sequences:
$A_Q = \{((u,q), (\mu_1(u,q),\ \mu_2(u,q), \ldots, \mu_k(u,q))): u \in U, q \in Q\}$, Where $\mu_i(u,q) \in I$ for all $i = 1, 2, \ldots k$.
The function $(\mu_1(u,q),\ \mu_2(u,q), \ldots, \mu_k(u,q))$ is called the membership function of multi Q-fuzzy set A_Q and $\mu_1(u,q) + \mu_2(u,q) + \cdots + \mu_k(u,q) \leq 1$, k is called the dimension of A_Q (Adam and Hassan, 2014a).
In other words, if the sequences of the membership functions have only k-terms (finite number of terms) the multi Q-fuzzy set is a function from $U \times Q$ to I^k such that for all $(u,q) \in U \times Q$, $\mu_{A_Q} = (\mu_1(u,q),\ \mu_2(u,q), \ldots, \mu_k(u,q))$. The set of all multi Q-fuzzy sets of dimension k in U and Q is denoted by $M^kQF(U)$

3. The Concept of α-cuts in Multi Q-fuzzy set

Definition 3.1 α-cuts in multi Q-fuzzy set
Let $A_Q \in M^kQF(U)$ and $\alpha \in [0,1]$. Then the α –cut of A_Q, denoted ${}^{\alpha}A_Q$ is defined as

$${}^{\alpha}A_Q = \left\{(u,q): \mu_{A_Q}(u,q) \geq \alpha\right\}.$$

The strong α –cut of A_Q, denoted ${}^{\alpha+}A_Q$ is defined as

$${}^{\alpha+}A_Q = \left\{(u,q): \mu_{A_Q}(u,q) > \alpha\right\}.$$

Theorem 3.2 Let $A_Q, B_Q \in M^kQF(U)$ and $\alpha \in [0,1]$. Then
(i) ${}^{\alpha+}A_Q \subseteq {}^{\alpha}A_Q$
(ii) If $\alpha_1 \leq \alpha_2 \Longrightarrow {}^{\alpha_2}A_Q \subseteq {}^{\alpha_1}A_Q$
(iii) ${}^{\alpha}(A_Q \cup B_Q) = {}^{\alpha}A_Q \cup {}^{\alpha}B_Q$
(iv) ${}^{\alpha}(A_Q \cap B_Q) = {}^{\alpha}A_Q \cap {}^{\alpha}B_Q$
(v) ${}^{\alpha+}(A_Q \cup B_Q) = {}^{\alpha+}A_Q \cup {}^{\alpha+}B_Q$
(vi) ${}^{\alpha+}(A_Q \cap B_Q) = {}^{\alpha+}A_Q \cap {}^{\alpha+}B_Q$

Proof

i). Observe from definition 3.1; α −cut always contains strong α −cut.

ii). From definition 3.1; observe that whenever $\alpha_1 \leq \alpha_2$ automatically ${}^{\alpha_1}A_Q$ will contain ${}^{\alpha_2}A_Q$.

iii). Let $(u,q) \in {}^{\alpha}(A_Q \cup B_Q) \Rightarrow \mu^i_{(A_Q \cup B_Q)}(u,q) \geq \alpha, i = 1,2,\ldots,k$

$$\Rightarrow \max\left[\mu^i_{A_Q}(u,q), \mu^i_{B_Q}(u,q)\right] \geq \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow \mu^i_{A_Q}(u,q) \geq \alpha \text{ or } \mu^i_{B_Q}(u,q) \geq \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow (u,q) \in {}^{\alpha}A_Q \text{ or } (u,q) \in {}^{\alpha}B_Q$$
$$\Rightarrow (u,q) \in {}^{\alpha}A_Q \cup {}^{\alpha}B_Q$$
$$\Rightarrow {}^{\alpha}(A_Q \cup B_Q) \subseteq {}^{\alpha}A_Q \cup {}^{\alpha}B_Q.$$

Suppose $(u,q) \in {}^{\alpha}A_Q \cup {}^{\alpha}B_Q$

$$\Rightarrow (u,q) \in {}^{\alpha}A_Q \text{ or } (u,q) \in {}^{\alpha}B_Q$$
$$\Rightarrow \mu^i_{A_Q}(u,q) \geq \alpha \text{ or } \mu^i_{B_Q}(u,q) \geq \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow \mu^i_{(A_Q \cup B_Q)}(u,q) \geq \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow (u,q) \in {}^{\alpha}(A_Q \cup B_Q)$$
$$\Rightarrow {}^{\alpha}A_Q \cup {}^{\alpha}B_Q \subseteq {}^{\alpha}(A_Q \cup B_Q).$$

Thus, the result follows.

iv). The proof follows similarly from (iii).

v). The proof follows similarly from (iv).

vi). Let $(u,q) \in {}^{\alpha+}\left(A_Q \cap B_Q\right) \Rightarrow \mu^i_{(A \cap B)_Q}(u,q) > \alpha, i = 1,2,\ldots,k$

$$\Rightarrow \min\left[\mu^i_{A_Q}(u,q), \mu^i_{B_Q}(u,q)\right] > \alpha, i = 1,2,\ldots,k$$

$\Rightarrow \mu^i_{A_Q}(u,q) > \alpha, i = 1,2,\ldots,k$ and

$$\mu^i_{B_Q}(u,q) > \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow (u,q) \in {}^{\alpha+}A_Q \text{ and } (u,q) \in {}^{\alpha+}B_Q$$
$$\Rightarrow (u,q) \in {}^{\alpha+}A_Q \cap {}^{\alpha+}B_Q$$
$$\Rightarrow {}^{\alpha+}(A_Q \cap B_Q) \subseteq {}^{\alpha+}A_Q \cap {}^{\alpha+}B_Q.$$

Also, let $(u,q) \in {}^{\alpha+}A_Q \cap {}^{\alpha+}B_Q$

$$\Rightarrow (u,q) \in {}^{\alpha+}A_Q \text{ and } (u,q) \in {}^{\alpha+}B_Q$$
$$\Rightarrow \mu^i_{A_Q}(u,q) > \alpha \text{ and } \mu^i_{B_Q}(u,q) > \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow \mu^i_{(A_Q \cap B_Q)}(u,q) > \alpha, i = 1,2,\ldots,k$$
$$\Rightarrow (u,q) \in {}^{\alpha+}(A_Q \cap B_Q)$$
$$\Rightarrow {}^{\alpha+}A_Q \cap {}^{\alpha+}B_Q \subseteq {}^{\alpha+}(A_Q \cap B_Q).$$

Hence, ${}^{\alpha+}A_Q \cup {}^{\alpha+}B_Q = {}^{\alpha+}(A_Q \cap B_Q)$.

Definition 3.3 Decomposition of Multi Q-fuzzy Soft Set

Let $U = \{u_1, u_2\}$, $Q = \{p, q, r\}$ and a multi Q fuzzy set A_Q over U and Q be $A_Q = \{\big((u_1,p), 0.3,0.2,0.5\big), \big((u_1,q), 0.2,0.8,0\big), \big((u_1,r), 0.1,0.5,0.3\big), \big((u_2,p), 0.3,0.1,0.2\big), \big((u_2,q), 0,0.3,0.7\big), \big((u_2,r), 0.2,0.3,0.1\big)\}$.

Let have the following distinct α-cuts defined by characteristic functions viewed as special membership functions:

$${}^{0.1}A_Q = \{\big((u_1,p),1,1,1\big),\big((u_1,q),1,1,0\big), \big((u_1,r),1,1,1\big),\big((u_2,p),1,1,1\big),\big((u_2,q),0,1,1\big), \big((u_2,r),1,1,1\big)\}, \alpha \geq 0.1.$$

$${}^{0.2}A_Q = \{\big((u_1,p),1,1,1\big),\big((u_1,q),1,1,0\big), \big((u_1,r),0,1,1\big),\big((u_2,p),1,0,1\big),\big((u_2,q),0,1,1\big), \big((u_2,r),1,1,0\big)\}, \alpha \geq 0.2.$$

$${}^{0.3}A_Q = \{\big((u_1,p),1,0,1\big),\big((u_1,q),0,1,0\big), \big((u_1,r),0,1,1\big),\big((u_2,p),1,0,0\big),\big((u_2,q),0,1,1\big), \big((u_2,r),0,1,0\big)\}, \alpha \geq 0.3.$$

$${}^{0.5}A_Q = \{\big((u_1,p),0,0,1\big),\big((u_1,q),0,1,0\big), \big((u_1,r),0,1,0\big),\big((u_2,p),0,0,0\big),\big((u_2,q),0,0,1\big), \big((u_2,r),0,0,0\big)\}, \alpha \geq 0.5.$$

$${}^{0.7}A_Q = \{\big((u_1,p),0,0,0\big),\big((u_1,q),0,1,0\big), \big((u_1,r),0,0,0\big),\big((u_2,p),0,0,0\big),\big((u_2,q),0,0,1\big), \big((u_2,r),0,0,0\big)\}, \alpha \geq 0.7.$$

$${}^{0.8}A_Q = \{\{\big((u_1,p),0,0,0\big),\big((u_1,q),0,1,0\big), \big((u_1,r),0,0,0\big),\big((u_2,p),0,0,0\big),\big((u_2,q),0,0,0\big), \big((u_2,r),0,0,0\big)\}, \alpha \geq 0.8.$$

Thus, converting each of the above α-cuts to a special Multi Q-fuzzy Set ${}_{\alpha}A_Q$ defined for each $(u,q) \in A_Q$ as

${}_{\alpha}A_Q = \alpha.\left({}^{\alpha}A_Q\right)$ ……………………………… (1)

We get

$${}_{0.1}A_Q = \{\big((u_1,p),0.1,0.1,0.1\big),\big((u_1,q),0.1,0.1,0\big), \big((u_1,r),0.1,0.1,0.1\big),\big((u_2,p),0.1,0.1,0.1\big), \big((u_2,q),0,0.1,0.1\big),\big((u_2,r),0.1,0.1,0.1\big)\},$$

$${}_{0.2}A_Q = \{\big((u_1,p),0.2,0.2,0.2\big),\big((u_1,q),0.2,0.2,0\big), \big((u_1,r),0,0.2,0.2\big),\big((u_2,p),0.2,0,0.2\big), \big((u_2,q),0,0.2,0.2\big),\big((u_2,r),0.2,0.2,0\big)\},$$

$${}_{0.3}A_Q = \{\big((u_1,p),0.3,0,0.3\big),\big((u_1,q),0,0.3,0\big), \big((u_1,r),0,0.3,0.3\big),\big((u_2,p),0.3,0,0\big), \big((u_2,q),0,0.3,0.3\big),\big((u_2,r),0,0.3,0\big)\},$$

$${}_{0.5}A_Q = \{\big((u_1,p),0,0,0.5\big),\big((u_1,q),0,0.5,0\big), \big((u_1,r),0,0.5,0\big),\big((u_2,p),0,0,0\big), \big((u_2,q),0,0,0.5\big),\big((u_2,r),0,0,0\big)\},$$

$${}_{0.7}A_Q = \{\big((u_1,p),0,0,0\big),\big((u_1,q),0,0.7,0\big),\big((u_1,r),0,0,0\big), \big((u_2,p),0,0,0\big),\big((u_2,q),0,0,0.7\big), \big((u_2,r),0,0,0\big)\},$$

$${}_{0.8}A_Q = \{\{\big((u_1,p),0,0,0\big),\big((u_1,q),0,0.8,0\big),\big((u_1,r),0,0,0\big), \big((u_2,p),0,0,0\big),\big((u_2,q),0,0,0\big), \big((u_2,r),0,0,0\big)\}.$$

It is easy to see that ${}_{0.1}A_Q \cup {}_{0.2}A_Q \cup {}_{0.3}A_Q \cup {}_{0.5}A_Q \cup {}_{0.7}A_Q \cup {}_{0.8}A_Q = A_Q$.

In other words, any Multi Q-fuzzy Set A_Q can be represented as the union of its special α-cuts ${}_{\alpha}A_Q$, and this representation is usually referred to as Decomposition of A_Q.

Moreover, if for each $(u,q) \in A_Q$, ${}_{\alpha+}A_Q$ is defined as

${}_{\alpha+}A_Q = \alpha.\left({}^{\alpha+}A_Q\right)$, ……………………………… (2)

we can see that by using a similar arguments, a Multi Q-fuzzy Set A_Q can be represented as the union of its special strong α-cuts ${}_{\alpha+}A_Q$, known as Decomposition of that Q-fuzzy Set.

Theorem 3.4 First Decomposition Theorem
Let $A_Q \in M^k QF(U)$, then $A_Q = \cup_{\alpha\in[0,1]}\ {}_\alpha A_Q$, where ${}_\alpha A_Q$ is as defined in (1).

Proof
For each $(u,q) \in A_Q$, let $y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$
Then for every $\alpha \in (y,1]$ we have $\mu^i_{A_Q}(u,q) = y < \alpha, i = 1,2,\dots,k$. Thus, ${}_\alpha A_Q = 0$.
On the other hand, for every $\alpha \in (0,y]$ we have $\mu^i_{A_Q}(u,q) = y \geq \alpha, i = 1,2,\dots,k$.
Thus, ${}_\alpha A_Q = \alpha$.
Hence, $(\cup_{\alpha\in[0,1]}\ {}_\alpha A_Q)\ (u,q) = \sup_{\alpha\in(0,y]} \alpha = y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$.

As the same argument is valid for each $(u,q) \in A_Q$, it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its α −cuts.

Theorem 3.5 Second Decomposition Theorem
Let $A_Q \in M^k QF(U)$, then $A_Q = \cup_{\alpha\in[0,1]}\ {}_{\alpha+}A_Q$, where ${}_{\alpha+}A_Q$ is as defined in (2).

Proof

The proof is analogous to that of the above theorem.
For each $(u,q) \in A_Q$, let $y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$. Then,
$(\cup_{\alpha\in[0,1]}\ {}_{\alpha+}A_Q)\ (u,q) = \sup_{\alpha\in(0,1]}\ {}_{\alpha+}A_Q =$
$\max[\sup_{\alpha\in(0,y]}\ {}^{\alpha+}A_Q, \sup_{\alpha\in(y,1]}\ {}^{\alpha+}A_Q]$.
Hence, $(\cup_{\alpha\in[0,1]}\ {}_{\alpha+}A_Q)\ (u,q) = \sup_{\alpha\in(0,y]} \alpha = y = \mu^i_{A_Q}(u,q), i = 1,2,\dots,k$.
As the same argument is valid for each $(u,q) \in A_Q$, it follows that each multi Q-fuzzy set can be uniquely represented as the family of all its strong α −cuts.

Conclusion
The idea of α-Cuts which was first applied to fuzzy set is extended to Multi Q-fuzzy set. It is shown among others that the α-Cut of the union of two multi Q-fuzzy set is the same as the union of their α-Cuts, and the α-Cut of the intersection of two multi Q-fuzzy set is the same as the intersection of their α-Cuts. It is further shown that, the same result is obtained with strong α-Cuts.

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