Science World Journal Vol. 14(No 3) 2019
www.scienceworldjournal.org
ISSN 1597-6343
Published by Faculty of Science, Kaduna State University

Full Length Research Article

BERNSTEIN LEAST-SQUARES TECHNIQUE FOR SOLVING FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

Oyedepo T.[1], Akinduko O.B.[2], Faniyi O.E.[3] and Okperhie E. P. [4]

[1]Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Nigeria.
[2]Mathematical Sciences Department, Adekunle Ajasin University, Akungba-Akoko, Ondo State, Nigeria.
[3]Department of Mathematics, Crown-hill University, Ilorin Kwara State, Nigeria
[4]Department of Mathematics, Federal Polytechnic, Bida, Niger State, Nigeria

Corresponding Author's Email Address: oyedepotaiye@yahoo.com

ABSTRACT
In this paper, a numerical technique for solving fractional Integro-Differential Equations (FIDEs) is presented. The fractional derivative is considered in the Caputo sense. The proposed method is Bernstein Least- Squares Technique (BLST) via Bernstein polynomials as basis functions. The suggested new technique reduced this type of problem to the solution of a system of linear algebraic equations and then solved using MAPLE 18. To demonstrate the accuracy and applicability of the presented method, some numerical problems are provided. Numerical results show that the method is easy to implement and accurate when applied to FIDEs. The graphical solution of the method is displayed.

Keywords: Bernstein polynomial; least-squares Technique

INTRODUCTION
Fractional calculus is a field dealing with integral and derivatives of arbitrary orders, and their applications in science, engineering and other fields. The idea is from the ordinary calculus. According to Leibniz [Adam, 2004; Caputo, 1967; Momani & Qaralleh, 2006; Samko *et al.*, 1993]. It was discovered by Leibniz in the year 1695 a few years he discovered ordinary calculus by later forgotten due to the complexity of the formula. Many real-world physical problems can be models by fractional integrodifferential equations e.g the modeling of the earthquake, reducing the spread of the virus, control the memory behavior of electric socket and many others. There are many fascinating or exciting books about fractional calculus and fractional differential equations (Caputo, 1967; Munkhammar, 2005; Samko *et al.*, 1993; Podlubny, 1999). Many FIDEs cannot be solved analytically, and hence finding good approximate solutions, using numerical techniques, will be very helpful. Several numerical methods to solve the FIDEs have been. The author in (Mittal & Nigam, 2008) applied the Adomian decomposition method (ADM) for the solution of FIDEs. Polynomial spline function was introduced in Rawashdeh (2006) for solving FIDEs. Cubic B-spline wavelets were introduced in Khowsrow *et al*. (2013) for the numerical solution of FIDEs. Mohamed *et al.* (2016) employed homotopy analysis transform method for solving FIDEs. Reference Taiwo *et al.* (2015) used Perturbed Chebyshev Polynomials for solving FIDEs. In their work, an approximate solution taken together with the Least - Squares method (LSM) is utilized to reduce the fractional Integra-differential equations to a system of algebraic equations, which are solved for the unknown constants associated with the approximate solution. Momani *et al.* (2006) applied an efficient method for finding the solution of systems of fractional integro-differential equations. Oyedepo et. al (2016) employed a method called numerical studies for solving fractional FIDEs using Least Squares Method and Bernstein Polynomials. The author in Oyedepo *et al.* (2019) applied Homotopy perturbation and LSM for solving FIDEs. Construction of orthogonal polynomials was introduced by Oyedepo *et al.* (2019) for the solution of FIDEs. Mohammed (2014) employed LSM for solving FIDEs using shifted Chebyshev polynomial of the first kind as the basis function. In other to improve on the existing methods in the literature, in this paper Bernstine Least-Squares Technique with the aid of Bernstein Polynomials is applied to solving FIDEs. The general form of the class of problem considered in this work is given as:

$$D^{\alpha}u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt, \quad o \leq x,t \leq 1, \tag{1}$$

With the following supplementary conditions:

$$u^{(j)}(0) = \delta_j \, j = 0,1,2,\dots,m-1, m-1 < \alpha \leq m, \; m \in, N \tag{2}$$

Where $D^{\alpha}u(x)$ indicates the $\propto th$ Caputo fractional derivative of $u(x);\ p(x), f(x)$,
$K(x,t)$ are given smooth functions, δ_j are real constant, x and t are real variables varying [0, 1] and $u(x)$ is the unknown function to be determined.

Some relevant basic definitions

Definition 1.
Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example, $D^{\frac{1}{2}}, D^{\pi}, D^{2+i}$ e.t.c

Definition 2.
Gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt \tag{3}$$

This integral converges when the real part of z is positive $(Re(z) \leq 0)$.

$$\Gamma(1+z) = z\Gamma(z) \tag{4}$$

When z is a positive integer

$$\Gamma(z) = (z-1)\,! \tag{5}$$

Definition 3.
Beta function is defined as

$$B(v,m) = \int_0^1 (1-u)^{v-1}u^{m-1}du = \frac{\Gamma(v)\Gamma(m)}{\Gamma(v+m)} =$$

$B(v,m)$,Where$v,m \in R_+$ (6)

Definition 4.

Riemann – Liouville fractional integral is defined as

$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(x)}{(x-t)^{1-\alpha}} dt,\ \alpha > 0, x > 0,$ (7)

J^{α} denotes the fractional integral of order $\propto$

Definition 5.

Riemann – Liouville fractional derivative denoted $D^{\propto}$is defined as

$D^{\propto} J^{\alpha} f(x) = f(x)$ (8)

Definition 6.

Riemann-Liouville fractional derivative defined as

$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\propto)} \int_0^x (x-s)^{n-\propto-1} f^n(s) ds,$ (9)

m is positive integer with the property that $m-1 <\propto< m$.

Definition 7.

The Caputor Factional Derivative is defined as

$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\propto)} \int_0^x (x-s)^{n-\propto-1} f^m(s) ds$ (10)

Where m is a positive integer with the property that $n-1 <\propto< n$

For example, if $0 <\propto< 1$ the caputo fractional derivative is

$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\propto)} \int_0^x (x-s)^{-\propto} f^1(s) ds$ (11)

Hence, we have the following properties:

(1) $J^{\alpha} J^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

(2) $J^{\alpha} x^{\gamma} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma,}\ \alpha > 0, \gamma > -1, x > 0$

(3) $J^{\alpha}\ D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^k(0) \frac{x^k}{k!}, \quad x > 0, n-1 < \alpha \leq n$

(4) $D^{\alpha} J^{\alpha} f(x) = f(x), \quad x > 0, n-1 < \alpha \leq n,$

(5) $D^{\alpha} C = 0, C$ is the constant,

(6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1.2, \ldots\}$

Definition 8.

Bernstein basis polynomials: A Bernstein polynomial of degree N is defined by

$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i} \quad i = 0,1 \ldots n,$ (12)

where,

$\binom{m}{i} = \frac{m!}{i!(m-1)!}$ (13)

Often, for mathematical convenience, we set $B_{i,m}(x) = 0$ if $< 0\ or\ j > m$

Definition 9.

Bernstein polynomials: A linear combination Bernstein basis polynomials

$u_m(x) = \sum_j^m a_j u_j(x)$ (14)

The Bernstein polynomial of degree n where $a_{j,}\ j = 0,1,2, \ldots \ldots$ are constants

Examples

The first few Bernstein basis polynomials are:

$u_0(x) = 1, u_1(x) = a_0(1-x) + a_1 x,\ u_2(x) = a_0(1-2x+x^2) + a_1(2x-2x^2) + a_2 x^2$

Definition 10

In this work, we defined absolute error as:

Absolute Error $= |U(x) - u_m(x)|; \quad 0 \leq x \leq 1,$ (15)

where $U(x)$ is the exact solution and $u_m(x)$ is the approximate solution.

Where $u_m(x)$ Bernstein polynomial of degree m wherea_j, j $= 0,1,2, \ldots$ are constants.

DEMONSTRATION OF THE PROPOSED METHOD

In this section, we demonstrated the two proposed methods mentioned above

Bernstein Least- Squares Technique (BLST)

The new technique via Bernstein polynomials as basis function is applied to find the numerical solution of fractional integrodifferential equation of the type in (1) and (2). This method is based on approximating the unknown function $u(x)$ by assuming an approximation solution of the form defined by (Rawashdeh, 2006).

Consider equation (1) operating with $J^{\propto}$ on both sides as follows:

$J^{\propto} D^{\alpha} u(x) = J^{\propto} f(x) + J^{\propto}(\int_0^x k(x,t) u(t) dt)$ (16)

$u(x) = \sum_{k=0}^{n-1} u^k(0) \frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^x k(x,t) u(t) dt]$ (17)

Substituting (14) into (17)

$\sum_j^m a_j u_j(x) = \sum_{k=0}^{n-1} u^k(0) \frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^x k(x,t) \sum_j^m a_j u_j(t)\, dt]$ (18)

Hence, the residual equation is obtained as

$R(a_{0,} a_1, \ldots \ldots \ldots, a_n) = \sum_j^m a_j u_j(x) - \{\sum_{k=0}^{n-1} u^k(0) \frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^x k(x,t) \sum_{j=0}^m a_j u_j(t)\, dt]\}$ (19)

Let

$S(a_{0,} a_1, \ldots \ldots, a_m) = \int_0^1 [R(a_{0,} a_1, \ldots \ldots, a_m)]^2 w(x) dx$ (20)

Where $w(x)$ is the positive weight function defined in the interval, [a, b]. In this work, we take $w(x) = 1$ for simplicity. Thus,

$S(a_{0,} a_1, \ldots \ldots \ldots, a_m) = \int_0^1 \left\{\sum_j^m a_j u_j(x) - \{\sum_{k=0}^{m-1} u^k(0) \frac{x^k}{k!} + J^{\propto} f(x) + [\int_0^x k(x,t) \sum_j^m a_j u_j(t)\, dt]\}\right\}^2 dx$ (21)

In order to minimize equation (22), we obtained the values of a_j $(j \geq 0)$ by finding the minimum value of S as:

$\frac{\partial S}{\partial a_j} = 0,\ j = 0,1,2 \ldots, m$ (22)

Applying (22) on (21), we have

$$\int_0^1 \left\{ \sum_{j=0}^m a_j u_j(x) - \{\sum_{k=0}^{n-1} u^k(0) \frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^x k(x,t) \sum_{i=0}^m a_j u_j(t)\, dt]\} \right\} dx$$

$\times \int_0^1 \{u_j^*(x) - J^{\propto}(\int_0^x k(x,t)u_j(t)dt)\}dx$ (23)

Thus, (23) are then simplified for $j = 0,1,\dots n$ to obtain $(m+1)$ algebraic system of equations in $(m+1)$ unknown a'_i s which are put in matrix form as follow:

$$A = \begin{pmatrix} \int_0^1 R(x,a_0)h_0dx & \int_0^1 R(x,a_1)h_0dx \cdots & \int_0^1 R(x,a_m)h_0dx \\ \int_0^1 R(x,a_0)h_1dx & \int_0^1 R(x,a_1)h_1dx \cdots & \int_0^1 R(x,a_m)h_1dx \\ \vdots & \vdots \ddots & \vdots \\ \int_0^1 R(x,a_0)h_mdx & \int_0^1 R(x,a_1)h_mdx \dots & \int_0^1 R(x,a_m)h_mdx \end{pmatrix},$$

$$B = \begin{pmatrix} \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{n-1} u^k(0)\frac{x^k}{k!}\right] h_0dx \\ \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{n-1} u^k(0)\frac{x^k}{k!}\right] h_1dx \\ \vdots \\ \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{n-1} u^k(0)\frac{x^k}{k!}\right] h_mdx \end{pmatrix} \quad (24)$$

Where

$$h_j = u_j^*(x) - J^{\propto}[\int_0^x k(x,t)u_j(t)dt], j = 0,1,\dots,m$$

$$R(x,a_j) = \sum_{i=0}^{m} a_i u_j(t) - J^{\propto}[\int_0^x k(x,t)\sum_{i=0}^{m} a_i u_j(t)\,dt], j = 0,1,\dots,m$$

The $(m+1)$ linear equations are then solved using maple 18 to obtain the unknown constants $a_j (j = 0(1)m)$, which are then substituted back into the assumed approximate solution to give the required approximation solution.

Numerical Examples

In this section, the technique discussed above is implemented on some problems. The problems are solved via Bernstein polynomials as basis functions. The problems are solved to illustrate the computational cost accuracy and efficiency of the proposed methods using Maple 18.

Example 1: Consider the following fractional Integro-differential

$$D^{\frac{3}{4}}u(x) = -\frac{x^2e^x}{5}u(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + e^x\int_0^x tu(t)dt \quad (25)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^3$

Applying BLST with the aid of Bernstein polynomials on (25) to get the exact solution as:

$$u(x) = x^3 \quad (26)$$

Example 2: Consider the following fractional Integro-differential

$$D^{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{2}x^3 + \int_0^x tu(t)dt \quad (27)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2$

Applying BLST with the aid of Bernstein polynomials on (27) to get the required approximate solution as:

$$u(x) = 1.74052882\times10^{-10}x^2 + 0.9999999990x^2 + 4.179314726\times10^{-10}x^3 - 1.479653948\times10^{-10} \quad (28)$$

Example 3: Consider the following fractional Integro-differential

$$D^{\frac{1}{2}}u(x) = (\cos(x) - \sin(x))\,u(x) + f(x) + \int_0^x x\sin(t)\,u(t)dt \quad (29)$$

$$f(x) = \frac{2x^{1.5}}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)}x^{0.5} + x(\cos(x) - x\sin(x) + x^2\cos(x)) \quad (30)$$

Subject to $u(0) = 0$. The exact solution is $U(x) = x^2 + x$

Applying BLST with the aid of Bernstein polynomials on (30) to get the required approximate solution as:

$$u(x) = -3.48\times10^{-8}x^3 + 1.000000052x^2 + 0.9999999810x + 1.410809629\times10^{-9} \quad (31)$$

Table 1: Numerical Results of Example 1

x	Exact Solution	Approximate Solution of New Technique	Approximate Solution of Method Rawashdeh (2006)	Absolute Error of New Technique	Absolute Error of Method Rawashdeh (2006)
0.0	0.000	0.00000000000000	0.000030	0.000E+00	3.000E-5
0.1	0.001	0.00100000000000	-	0.000E+00	-
0.2	0.008	0.00800000000000	0.0080371	0.000E+00	3.710E-5
0.3	0.273	0.02700000000000	-	0.000E+00	-
0.4	0.064	0.06400000000000	0.064024	0.000E+00	2.400E-5
0.5	0.125	0.12500000000000	-	0.000E+00	-
0.6	0.216	0.21600000000000	0.216084	0.000E+00	8.400-5
0.7	0.343	0.34300000000000	-	0.000E+00	-
0.8	0.512	0.51200000000000	0.512043	0.000E+00	4.300E-5
0.9	0.729	0.72900000000000	-	0.000E+00	-
1.0	1.000	1.00000000000000	1.000028	0.000E+00	2.800E-5

Table 2: Numerical Results of Example 2

x	Exact Solution	Approximate Solution of New Technique	Approximate Solution of Method Mohamed et. al (2016)	Absolute Error of New Technique	Absolute Error of Method Mohamed et. al (2016)
0.0	0.00	-0.00000000014797	0.0000000000000	1.480E-10	0.000E+00
0.1	0.01	0.00999999988200	0.00999098347500	1.160E-10	9.017E-06
0.2	0.04	0.03999999989000	0.03984425053000	1.030E-10	1.557E-04
0.3	0.09	0.08999999989000	0.08915051900000	1.079E-10	8.495E-04
0.4	0.16	0.15999999990000	0.15711315420000	1.672E-10	2.887E-03
0.5	0.25	0.24999999990000	0.24243543660000	2.196E-10	7.565E-03
0.6	0.36	0.35999999980000	0.34319333660000	2.857E-10	1.681E-02
0.7	0.49	0.48999999980000	0.45669292160000	2.710E-10	3.331E-02
0.8	0.64	0.63999999970000	0.57931151210000	3.645E-10	6.069E-02
0.9	0.81	0.80999999960000	0.70632167110000	4.550E-10	1.037E-01
1.0	1.00	0.99999999950000	0.83169710000000	5.560E-10	1.683E-01

Table 3: Numerical Results of Example 3

x	Exact Solution	Approximate Solution New Techniques	Approximate Solution of Method Rawashdeh (2006)	Absolute Error of New Technique	Absolute Error of Method Rawashdeh (2006)
0.0	0.000	0.00000000141081	0.000286	1.411E-09	0.000E+00
0.1	0.110	0.11000000000000	-	3.990E-12	-
0.2	0.240	0.23999999940000	0.240010	5.876E-10	1.000E-5
0.3	0.390	0.38999999940000	-	5.488E-10	-
0.4	0.560	0.55999999990000	0.560087	9.639E-11	8.700E-5
0.5	0.750	0.75000000050000	-	5.608E-10	-
0.6	0.960	0.96000000120000	0.960053	1.214E-09	5.300E-5
0.7	1.190	1.19000000100000	-	1.654E-09	-
0.8	1.440	1.44000000100000	1.440081	1.673E-09	8.100E-05
0.9	1.710	1.71000000100000	-	1.062E-09	-
1.0	2.000	1.99999999900000	2.000617	3.892E-10	6.170E-5

Graphical Representation of the Method

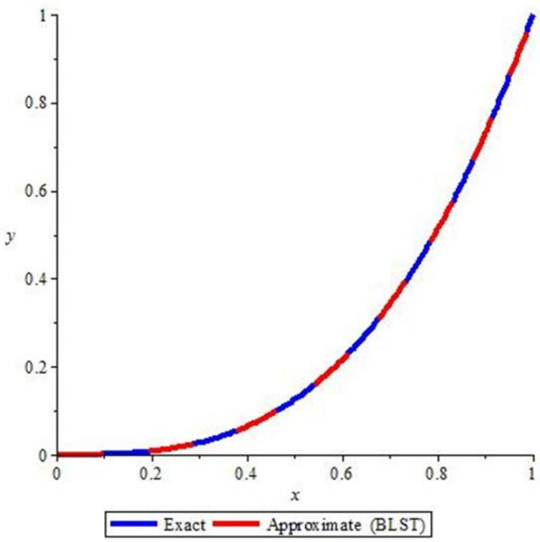

Figure 1: Showing the graph of approximation solution and exact solution of example 1

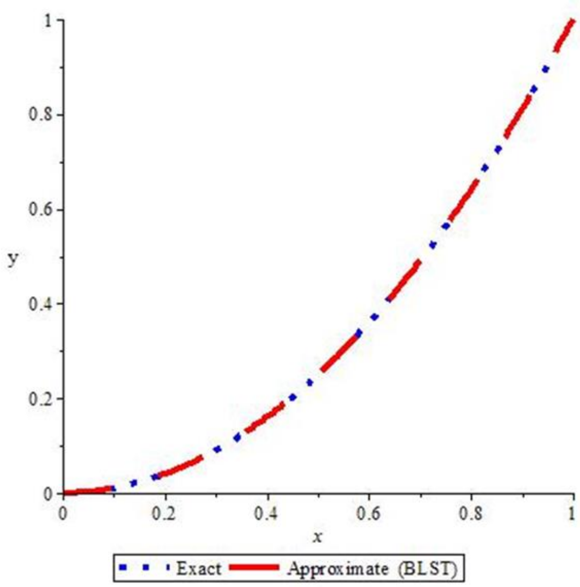

Figure 2: Showing the graph of approximation solution and exact solution of example 2

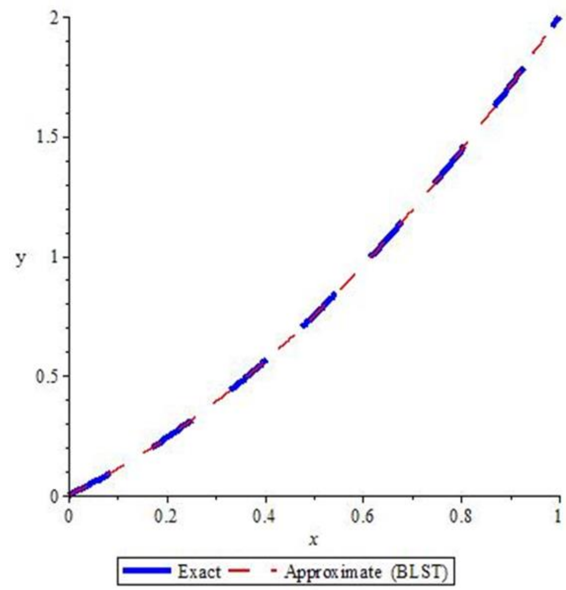

Figure 3: Showing the graph of approximation solution and exact of example 3

DISCUSSION

All the problems presented in this study were solved using maple 18. Table 1 for problem 1 shows that the new technique in this study is more accurate than the method of Rawashdeh (2006). Table 2 for example 2 reveals the new technique via the Bernstein Polynomial as basis function is more accurate than the method of Mohamed *et al*. (2016). Also Table 3 for example 3, a comparison was made with the method of Rawashdeh (2006), where again the new technique was seen to be better in terms of accuracy. It is to be noted that these comparisons were made for only those values that are available in the existing literature. The graphs in figures 1 − 3 are presented to further buttress the above observation. However, it was clear that errors of the new method are smaller than that of Rawashdeh (2006) and Mohamed *et al*. (2016)

Conclusion

The study applied the new technique via Bernstein polynomial as basis functions to find the solution of FIDEs. Some problems were solved using the BLST. The results obtained compared with Rawashdeh (2006) and Mohamed *et al*. (2016) showed that BLST is more accurate than Rawashdeh (2006) and Mohamed *et al*. (2016). Hence, calculation showed that BLST is a powerful and efficient technique in finding a very good solution for this type of equation. Also, the results were presented in graphical forms to further demonstrate the method.

REFERENCES

Adam, L. (2004): Fractional Calculus: history, definition, and application for the engineer. *Department of Aerospace and Mechanical Engineering University of Notre Dame*.

Caputo, M. (1967): Linear models of dissipation whose Q is almost frequency Independent. *Geophysical Journal International*, 13(5):529 – 539.

Khowsrow M., Monireh N.S & Azadeh O. (2013). Numerical solution of fractional integrodifferential equation by using cubic B-spline wavelets. *Proceeding of the World Congress on Engineering U .K*, **1**, 3-5

Mittal, R. C., and Nigam, R. (2008): Solution of Fractional Integro-differential Equations by Adomian Decomposition Method. *International Journal of Applied Mathematics and Mechanics,* 4(2), 87-94.

Munkhammar, J. D. (2005): Fractional calculus and the Taylor Riemann series. *Undergraduate Mathematics Journal*, 5(1): 1 - 19.

Mohamed, S, Muteb R. and Refah A. (2016). Solving fractional integrodifferential equations by homotopy analysis transform method. *International Journal of pure and Applied Mathematics,* 106(4), 1037 - 1055

Mohammed, D.Sh. (2014): Numerical solution of fractional integro-differential equations by least square method and shifted chebyshev polynomial. *Mathematics Department, Faculty of science,Zagazig University, Zagazig, Egypt.*

Momani, S. and Qaralleh A. (2006): An efficient method for solving systems ofFractional Integro-differential equations. *Computer and Mathematica with Applications* vol.25, 459-570.

Oyedepo, T., Taiwo, O.A., Abubakar J. U., & Ogunwobi, Z.O. (2016): Numerical Studies for Solving Fractional Integro-Differential Equations by using Least Squares

Method and Bernstein Polynomials. *Fluid Mechanics Open Access* 3(3): 1-7 Oyedepo, T., Taiwo, O.A., Adebisi A.F., Ishola C.Y. & Faniyi, O.E. (2019): Least Squares Method and Homotopy Perturbation Method for Solving Fractional Integro-Differential Equations. *Pasifi Journal Of Science and Technology, 20(1), 86 – 95.*

Oyedepo, T. & Taiwo, O.A. (2019): Numerical Studies for Solving Linear Fractional Integro-diferential Equation Based on Constructed Orthogonal Polynomials. *Journal of Science, Technology* ***&*** *Education, 7(1), 1 - 13.*

Podlubny, I.(1999) :*Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. New York: Academic Press. 19

Rawashdeh, E. A (2006): Numerical solution of fractional integro-differential equations by collocation method. *Appl Math Comput*. 176 .1-6

Samko, S.G., Kilbas, A.A. and Marichev O. I (1993): *Fractional Integrals and Derivatives.* Theory and Applications, Gordon and Breach, Yverdon.

Taiwo, O. A. & Fesojaye M. O. (2015). Perturbation Least-Squares Chebyshev method for solving fractional order integro-differential equations. *Theoretical Mathematics and Applications*, 5(4), 37 – 47