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SOME ALGEBRAIC STRUCTURES OF MULTI-FUZZY SET

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ABSTRACT
The paper presents the concept of multi-fuzzy set and its operations. In addition some algebraic structures of multi-fuzzy set were introduced; in particular, the lattice structures, semigroups and groupoids of multi-fuzzy set were established. Some related results were also provided.

Keywords: Multi-fuzzy set, Groupoid, Semigroup, Lattice

1. INTRODUCTION

A fuzzy set which is a generalized set of objects occurring with a continuum of degrees (grades) of membership was introduced by Zadeh (1965) to model vagueness and other loose concepts. An extensive representation of fuzzy sets was found in (Zadeh, 1965; Klir and Folger, 1987). Yager (1986) initiated the theory of fuzzy multisets which model the case where indistinguishable objects possess a particular property to a certain degree. Moreover, if we go the other way and fuzzify the number of occurrences of each object, then we get a multi-fuzzy set which was introduced in order to fuzzify the basic property of multisets and to further define multisets where the number of occurrences forms a fuzzy set (Blizard, 1989). For an overview and applications of the theories of multisets, fuzzy multisets and mult-fuzzy sets refer to (Blizard, 1991; Kosko, 1992; Singh et al., 2008; Isah and Tella, 2015; Singh and Isah, 2016; Isah, 2019). As certain algebraic structures have found applications in formal language theory, computer science, sequential machines etc., (Knuth, 1981; Tremblay and Manohar, 1997; Priss and Old, 2006; Singh and Isah, 2015), this paper aimed to introduce some algebraic structures of multi-fuzzy sets.

2. Preliminaries

2.1 Fuzzy Sets [Zadeh, 1965; Klir and Yuan, 1995]

Definition 2.1.1 A fuzzy set (class) $\tilde{A}$ in X is characterized by a membership function $\mu_{\check{A}}(x)$ which associates with each point x in X, a real number $\mu_{\check{A}}(x)$ in the interval $[0,1]$. The value of $\mu_{\check{A}}(x)$ represents the grade of membership of x in $\check{A}$.

Let $\check{A}$ and $\check{B}$ be two fuzzy sets, then

(a) $\check{A} \subseteq \check{B}$ if and only if
$\mu_{\check{A}}(x) \le \mu_{\check{B}}(x), \quad \forall x \in X$.

(b) $\check{A} \cup \check{B} = \check{C}$ such that
$\mu_{\check{C}}(x) = \max[\mu_{\check{A}}(x), \mu_{\check{B}}(x)], \forall x \in X$.

(c) $\check{A} \cap \check{B} = \check{C}$ such that
$\mu_{\check{C}}(x) = \min[\mu_{\check{A}}(x), \mu_{\check{B}}(x)], \forall x \in X$.

Example 2.1.2
Let $\check{A} = \{(x, 0.1), (y, 0.4), (z, 0.7)\}$ and $\check{B} = \{(x, 0.6), (y, 0.3), (z, 0.2)\}$ be fuzzy sets, then $\check{A} \cup \check{B} = \{(x, 0.6), (y, 0.4), (z, 0.7)\}$ and $\check{A} \cap \check{B} = \{(x, 0.1), (y, 0.3), (z, 0.2)\}$.

2.2 Multisets [Blizard, 1991; Singh et al., 2008]

Definition 2.2.1 Let $X = \{x_1, x_2, x_3, \cdots, x_j, \cdots\}$ be set. A multiset or mset A over X is a cardinal-valued function i.e., $A: X \to N = \{0,1,2,\cdots\}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object *x* occurs in A.

Let A be a multiset containing one occurrence of *a*, two occurrences of *b*, and three occurrences of *c*, then A can be represented as $A = [[a, b, b, c, c, c]] = [a, b, b, c, c, c] = [a, b, c]_{1,2,3} = [a, 2b, 3c] = [a.1, b.2, c.3] = [1/a, 2/b, 3/c] = [a^1, b^2, c^3] = [a^1\ b^2\ c^3]$. For convenience, the curly brackets are also used in place of the square brackets.

Definition 2.2.2 Let A and B be two msets over a given domain set X. Then

(a) $A \subseteq B$ if $m_A(x) \le m_B(x), \forall x \in X$.
(b) $A = B$ if $m_A(x) = m_B(x), \forall x \in X$.
(c) $A \cup B = \max[m_A(x), m_B(x)], \forall x \in X$.
(d) $A \cap B = \min[m_A(x), m_B(x)], \forall x \in X$.
(e) $A + B = A \uplus B = m_A(x) + m_B(x), \forall x \in X$.
(f) $A - B = \max[m_A(x) - m_B(x), 0], \forall x \in X$.

Example 2.2.3
Let $A = [7/a, 4/b, 3/c]$ and $B = [1/a, 3/b, 2/c]$ be msets over $X = \{a, b, c\}$, then
$A \cup B = [7/a, 4/b, 3/c]$, $A \cap B = [1/a, 3/b, 2/c]$, $A + B = [8/a, 7/b, 5/c]$ and
$$A - B = [6/a, 1/b, 1/c].$$

2.3 Multi-fuzzy sets [Syropoulos, 2006]

Definition 2.3.1 Let X be a (fixed) universe, then a multi-fuzzy set A is a function $A: X \to N_0 \times I$, where N_0 is the set of all positive integers including zero and $I = [0,1]$. Thus $A(x) = (n, i)$ denotes that the degree to which x occurs n times in the multi-fuzzy set is equal to i.

Let A be a multi-fuzzy set, then

(a) the multiplicity function is defined as $A_m(x): X \to N_0$ and
(b) the membership function is defined as $\mu_A(x): X \to I$.

Obviously, if $A(x) = (n, i)$, then $A_m(x) = n$ and $\mu_A(x) = i$.

Definition 2.3.2 Let A and B be two multi-fuzzy sets over X, then Sub multi-fuzzy set:

$A \subseteq B$ if $A_m(x) \leq B_m(x)$ and $\mu_A(x) \leq \mu_B(x), \forall x \in X$.
Their Equality:
$A = B$ if $A_m(x) = B_m(x)$ and $\mu_A(x) = \mu_B(x), \forall x \in X$.
Their Union is
$(A \cup B)(x)$
$= (\max\{A_m(x), B_m(x)\}, \max\{\mu_A(x), \mu_B(x)\}), \forall x \in X$.
Their Intersection is
$(A \cap B)(x)$
$= (\min\{A_m(x), B_m(x)\}, \min\{\mu_A(x), \mu_B(x)\}), \forall x \in X$.
Their Sum is
$(A \uplus B)(x) = (A_m(x) + B_m(x), (\mu_A + \mu_B)(x)), \forall x \in X$,
where $(\mu_A + \mu_B)(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\,\mu_B(x)$.
Their Difference:
$A \ominus B = (\max\{A_m(x) - B_m(x), 0\}, \min\{\mu_A(x), \mu_B(x)\}), \forall x \in X$.
Their Symmetric Difference:
$A \triangle B = (|A_m(x) - B_m(x)|, |\mu_A(x) - \mu_B(x)|), \forall x \in X$.

Example 2.3.3
Let $A = \{(3,0.5)/x, (5,0.7)/y, (7,0.4)/z\}$ and $B = \{(2,0.2)/x, (4,0.8)/y, (5,0.3)/z\}$ be two Multi-fuzzy sets over $X = \{x, y, z\}$. Then

$$A \cup B = \{(3,0.5)/x, (5,0.8)/y, (7,0.4)/z\}$$
$$A \cap B = \{(2,0.2)/x, (4,0.7)/y, (5,0.3)/z\}$$
$$A \uplus B = \{(5,0.6)/x, (9,0.94)/y, (12,0.58)/z\}.$$
$$A \ominus B = \{(1,0.2)/x, (1,0.7)/y, (2,0.3)/z\}$$
$$A \triangle B = \{(1,0.3)/x, (1,0.1)/y, (2,0.1)/z\}$$

3. Some Algebraic Structures of Multi-fuzzy Sets

Proposition 3.1 Let L be the collection of all multi-fuzzy sets over X, and let $A, B, C \in L$ then
$(L, \cup)$ is a commutative, idempotent semigroup.

Proof
Let $A \in L$ and $x \in X$, then $(A \cup A)(x) = (max\{A_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x)) = A(x)$
i.e., $(L, \cup)$ is idempotent.

Let $A, B \in L$ and $x \in X$, then $(A \cup B)(x) = (max\{A_m(x), B_m(x)\}, max\{\mu_A(x), \mu_B(x)\})$
$=$
$(max\{B_m(x), A_m(x)\}, max\{\mu_B(x), \mu_A(x)\})$
$= (B \cup A)(x)$
i.e., $(L, \cup)$ is commutative.

Let $A, B, C \in L$ and $x \in X$, then $(A \cup (B \cup C))(x) = (max\{A_m(x), max\{B_m(x), C_m(x)\}\},$
$max\{\mu_A(x), max\{\mu_B(x), \mu_C(x)\}\})$
$= (max\{A_m(x), B_m(x), C_m(x)\},$
$max\{\mu_A(x), \mu_B(x), \mu_C(x)\})$
$= (\max\{max\{A_m(x), B_m(x)\}, C_m(x)\},$
$max\{max\{\mu_A(x), \mu_B(x)\}, \mu_C(x)\})$
$((A \cup B) \cup C)(x)$
i.e., $(L, \cup)$ is associative.
Thus, $(L, \cup)$ is commutative, idempotent semigroup.

Proposition 3.2 Let L be the collection of all multi-fuzzy sets over X, and let $A, B, C \in L$ then
$(L, \cap)$ is a commutative, idempotent semigroup.

Proof
Let $A \in L$ and $x \in X$, then
$(A \cap A)(x) = (min\{A_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x)) = A(x)$
i.e., $(L, \cap)$ is idempotent.

Let $A, B \in L$ and $x \in X$, then
$(A \cap B)(x) = (min\{A_m(x), B_m(x)\}, min\{\mu_A(x), \mu_B(x)\})$
$= (min\{B_m(x), A_m(x)\}, min\{\mu_B(x), \mu_A(x)\})$
$= (B \cap A)(x)$
i.e., $(L, \cap)$ is commutative.

Let $A, B, C \in L$ and $x \in X$, then
$(A \cap (B \cap C))(x) = (min\{A_m(x), min\{B_m(x), C_m(x)\}\},$
$min\{\mu_A(x), min\{\mu_B(x), \mu_C(x)\}\})$
$= (min\{A_m(x), B_m(x), C_m(x)\},$
$min\{\mu_A(x), \mu_B(x), \mu_C(x)\})$
$= (\min\{min\{A_m(x), B_m(x)\}, C_m(x)\},$
$min\{min\{\mu_A(x), \mu_B(x)\}, \mu_C(x)\})$
$((A \cap B) \cap C)(x)$
i.e., $(L, \cap)$ is associative.
Hence, $(L, \cap)$ is commutative, idempotent semigroup.

Proposition 3.3 Let L be the collection of all multi-fuzzy sets over X, and let $A, B \in L$ then
the following hold.
i. $(A \cup B) \cap A = A$
ii. $(A \cap B) \cup A = A$

Proof
i. Let $A, B \in L$ and $x \in X$, then $((A \cup B) \cap A)(x) = (min\{\max\{A_m(x), B_m(x)\}, A_m(x)\},$
$min\{\max\{\mu_A(x), \mu_B(x)\}, \mu_A(x)\})$
if $A_m(x) < B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$((A \cup B) \cap A)(x)$
$= (min\{B_m(x), A_m(x)\}, min\{\mu_B(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x))$
$= A(x)$.

if $A_m(x) < B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$((A \cup B) \cap A)(x)$
$= (min\{B_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x))$
$= A(x)$.
if $A_m(x) > B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$((A \cup B) \cap A)(x)$
$= (min\{A_m(x), A_m(x)\}, min\{\mu_B(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x))$
$= A(x)$.
if $A_m(x) > B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$((A \cup B) \cap A)(x)$
$= (min\{A_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$
$= (A_m(x), \mu_A(x))$
$= A(x)$.
if $A_m(x) = B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$((A \cup B) \cap A)(x)$
$= (min\{A_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$

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$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$

if $A_m(x) = B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$$\big((A \cup B) \cap A\big)(x)$$
$$= (min\{ A_m(x), A_m(x)\}, min\{\mu_B(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
if $A_m(x) = B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$$\big((A \cup B) \cap A\big)(x)$$
$$= (min\{ A_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
if $A_m(x) < B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$$\big((A \cup B) \cap A\big)(x)$$
$$= (min\{ B_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
if $A_m(x) > B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$$\big((A \cup B) \cap A\big)(x)$$
$$= (min\{ A_m(x), A_m(x)\}, min\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
Thus, $(A \cup B) \cap A = A$.

(ii)$\big((A \cap B) \cup A\big)(x) =$
$$(max\{\min\{A_m(x), B_m(x)\}, A_m(x)\},$$
$$max\{\min\{\mu_A(x), \mu_B(x)\}, \mu_A(x)\})$$
if $A_m(x) < B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) < B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_B(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) > B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ B_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) > B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ B_m(x), A_m(x)\}, max\{\mu_B(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) = B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) = B_m(x)$ and $\mu_A(x) < \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) = B_m(x)$ and $\mu_A(x) > \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_B(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) < B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ A_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
If $A_m(x) > B_m(x)$ and $\mu_A(x) = \mu_B(x)$ we have
$$\big((A \cap B) \cup A\big)(x)$$
$$= (max\{ B_m(x), A_m(x)\}, max\{\mu_A(x), \mu_A(x)\})$$
$$= (A_m(x), \mu_A(x))$$
$$= A(x).$$
Hence, $(A \cap B) \cup A = A$.

Proposition 3.4 Let L be the collection of all multi-fuzzy sets over X, and let $A, B, C \in L$ then

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

(i) Let $x \in X$ and $A, B, C \in L$, then $(A \cup (B \cap C))(x) =$
$$(max\{A_m(x), min\{B_m(x), C_m(x)\}\},$$
$$max\{\mu_A(x), min\{\mu_B(x), \mu_C(x)\}\})$$

$$= (\min\{\max\{A_m(x), B_m(x)\},$$
$$max\{A_m(x), C_m(x)\}\}),$$
$$(\min\{\max\{\mu_A(x), \mu_B(x)\},$$
$$max\{\mu_A(x), \mu_C(x)\}\}),$$
$$= (\min\{\max\{A_m(x), B_m(x)\},$$
$$\max\{\mu_A(x), \mu_B(x)\}\}),$$
$$(\min\{\max\{\{A_m(x), C_m(x)\},$$
$$\max\{\mu_A(x), \mu_C(x)\},$$
$$= (A \cup B) \cap (A \cup C))(x)$$
i.e., $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) $(A \cap (B \cup C))(x) =$
$$(min\{A_m(x), max\{B_m(x), C_m(x)\}\},$$
$$min\{\mu_A(x), max\{\mu_B(x), \mu_C(x)\}\})$$
$$= (\max\{\min\{A_m(x), B_m(x)\},$$
$$min\{A_m(x), C_m(x)\}\}),$$
$$(\max\{\min\{\mu_A(x), \mu_B(x)\},$$
$$min\{\mu_A(x), \mu_C(x)\}\}),$$
$$= (\max\{\min\{A_m(x), B_m(x)\},$$
$$\min\{\mu_A(x), \mu_B(x)\}\}),$$
$$(\max\{\min\{\{A_m(x), C_m(x)\},$$
$$\min\{\mu_A(x), \mu_C(x)\}$$
$$= (A \cap B) \cup (A \cap C))(x)$$
i.e., $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proposition 3.5 The structure $(L, \cup, \cap)$ is a bounded distributive lattice.

Proof

By propositions 3.1, 3.2, 3.3 and 3.4, $(L, \cup, \cap)$ is a bounded distributive lattice with the empty and universal multi-fuzzy sets as the lower and upper bounds, respectively.

Proposition 3.6 Let A be a multi-fuzzy set over X and $P(A)$ be the power multi-fuzzy set of A, then $(P(A),\cup,\cap)$ is a sub lattice of $(L,\cup,\cap)$ with empty multi-fuzzy set and A as the lower and upper bounds, respectively.

Proposition 3.7 Let A, B, C be multi-fuzzy sets over X, then $(L, \uplus)$ is a commutative semigroup.

Proof
Let $A, B \in L$ and $x \in X$, then

$$\begin{aligned}(A \uplus B)(x) &= (A_m(x) + B_m(x), \mu_A(x) + \mu_B(x)) \\ &= (B_m(x) + A_m(x), \mu_B(x) + \mu_A(x)) \\ &= (B \uplus A)(x)\end{aligned}$$

i.e., $(L, \uplus)$ is commutative.

Let $A, B, C \in L$ and $x \in X$, then

$$\begin{aligned}(A \uplus (B \uplus C))(x) &= (A_m(x) + (B_m(x) + C_m(x)), \mu_A(x) \\ &\quad + (\mu_B(x) + \mu_B(x))) \\ &= ((A_m(x) + B_m(x)) + C_m(x), (\mu_A(x) + \mu_B(x)) \\ &\quad + \mu_B(x)) \\ &= ((A \uplus B) \uplus C)(x)\end{aligned}$$

i.e., $(L, \uplus)$ is associative.
Hence, $(L, \uplus)$ is a commutative semigroup.

Remark 3.8
Proposition 3.1 fails for $(L, \uplus)$.
Counter Example
let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}$, then

$$A \uplus A = \{(6,0.75)/x, (10,0.91)/y, (4,0.64)/z\} \neq A.$$

Remark 3.9
Proposition 3.3 fails for $(L, \uplus)$.
Counter Example
Let $A = \{(13,0.1)/x, (4,0.2)/y, (7,0.6)/z\}$ and $B = \{(3,0.3)/x, (14,0.8)/y, (1,0.1)/z\}$ be two Multi-fuzzy sets over $X = \{x, y, z\}$. Then

$$\begin{aligned}A \uplus B &= \{(16,0.37)/x, (18,0.84)/y, (8,0.64)/z\}. \\ (A \uplus B) \cap A &= \{(16,0.37)/x, (18,0.84)/y, (8,0.64)/z\} \\ &\quad \cap \{(13,0.1)/x, (4,0.2)/y, (7,0.6)/z\} \\ &= \{(13,0.1)/x, (4,0.2)/y, (7,0.6)/z\} \neq A.\end{aligned}$$

Proposition 3.10 Let L be the collection of all multi-fuzzy sets over X, the $(L, \ominus)$ is a non-commutative, non-idempotent groupoid.

Proof
Counter Example
let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}$, then

$$A \ominus A = \{(0,0.5)/x, (0,0.7)/y, (0,0.4)/z\} \neq A.$$

Therefore, $(L, \ominus)$ is non-idempotent.

let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}$ and then $B = \{(4,0.6)/x, (8,0.3)/y, (9,0.2)/z\}$

$$\begin{aligned}A \ominus B &= \{(0,0.5)/x, (0,0.3)/y, (0,0.2)/z\} \\ \neq B \ominus A &= \{(1,0.5)/x, (3,0.3)/y, (7,0.2)/z\}\end{aligned}$$

Therefore, $(L, \ominus)$ is non-commutative.
Moreover, for any $A, B \in L$, $A \ominus B \in L$.
Hence, $(L, \ominus)$ is non-idempotent, non-commutative groupoid.

Proposition 3.11 Let L be the collection of all multi-fuzzy sets over X, then $(L, \triangle)$ is a commutative groupoid.

Proof
Let $A, B \in L$ and $x \in X$, then

$$\begin{aligned}(A \triangle B)(x) &= (|A_m(x) - B_m(x)|, |\mu_A(x) - \mu_B(x)|) \\ &(|B_m(x) - A_m(x)|, |\mu_B(x) - \mu_A(x)|) \\ &= (B \triangle A)(x)\end{aligned}$$

i.e., $(L, \triangle)$ is commutative.
Moreover, for all $A, B \in L, A \triangle B \in L$.
Hence, $(L, \triangle)$ is a commutative groupoid.

Remark 3.12
$(L, \triangle)$ is non-idempotent.

Counter Example
let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}$, then

$$A \triangle A = \{(0,0)/x, (0,0)/y, (0,0)/z\} \neq A.$$

Remark 3.13
$(L, \triangle)$ is non-associative.

Counter Example
let $A = \{(3,0.5)/x, (5,0.7)/y, (2,0.4)/z\}, B = \{(4,0.6)/x, (8,0.3)/y, (9,0.2)/z\}$ and

$$C = \{(1,0.5)/x, (3,0.3)/y, (7,0.2)/z\}$$

$$\begin{aligned}A \triangle B &= \{(1,0.1)/x, (3,0.4)/y, (7,0.2)/z\} \\ (A \triangle B) \triangle C &= \{(0,0.4)/x, (0,0.1)/y, (1,0)/z\} \\ B \triangle C &= \{(3,0.1)/x, (5,0)/y, (2,0)/z\} \\ A \triangle (B \triangle C) &= \{(0,0.4)/x, (0,0.7)/y, (0,0.4)/z\}\end{aligned}$$

Thus,

$$(A \triangle B) \triangle C \neq A \triangle (B \triangle C).$$

Conclusion
Using multi-fuzzy set operations, some algebraic structures were introduced. It is established that, as the class of all multi-fuzzy set over a set together with the operations of union and intersection formed a bounded distributive lattice, the power multi-fuzzy set with similar operations formed its sub lattice. Other operations were also used to form groupoids and monoids. The constructions made in this paper are useful tools in mathematics, computer science and other sciences etc.

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