

APPLICATION OF THE q -HOMOTOPY ANALYSIS TRANSFORM METHOD (q -HATM) TO THE SOLUTION OF A FRACTIONAL ATTRACTION KELLER-SEGEL CHEMOTAXIS MODEL

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ABSTRACT

The main purpose of this paper is to construct approximate analytic solutions for a time-fractional attraction Keller-Segel (TF-AKS) chemotaxis model using the q -homotopy analysis transform method (q -HATM). The obtained results and numerical simulations for three sets of initial data describe the behavior of the system. This further assert the convenience, computational efficiency and wide applicability of the proposed method even to more complex coupled systems of partial differential equations arising from mathematical biology.

Keywords: Keller-Segel chemotaxis model, Caputo derivative, Laplace transform, q -homotopy analysis method

INTRODUCTION

Obtaining exact solutions of fractional differential equations appear to be more difficult than their classical integer-order counterparts. Hence, a lot of attention have been devoted to develop very effective semi-analytical and numerical techniques for finding approximate solutions to this class of problems. Some of these techniques include the Adomian decomposition method (ADM) (Momani, 2005; Momani *et al.*, 2006), Laplace decomposition method (LDM) (Jafari *et al.*, 2011; Khan *et al.*, 2011), Homotopy analysis method (HAM) (Liao, 1992; Liao, 2003; Zurigat *et al.*, 2010), Homotopy perturbation method (HPM) (Momani *et al.*, 2008) and Variational iteration method (VIM) (Jafari *et al.*, 2012). Another very powerful technique is the q -homotopy analysis transform method (q -HATM) (Kumar *et al.*, 2017; Prakasha *et al.*, 2017; Singh *et al.*, 2019). It combines the traditional q -homotopy analysis method (q -HAM) due to El-Tawil and Huseen (El-Tawil *et al.*, 2012; El-Tawil *et al.*, 2013) with the Laplace transform method (LTM) to simplify computational procedures without any need for discretization or restrictive assumptions. The q -HAM extends the classical homotopy analysis method (HAM) by incorporating a parameter $q \in [0, \frac{1}{n}]$, $n \geq 1$. The presence of the term $(\frac{1}{n})^m$ in the q -HATM solution ensures faster convergence than the classical HAM. The central focus of this paper is to employ the q -HATM to construct approximate series solutions for the following one-dimensional time-fractional Keller-Segel chemotaxis model (TF-AKS):

$$\begin{cases} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = d_u \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial}{\partial x} \left(u(x, t) \frac{\partial \chi(v(x, t))}{\partial x} \right) \\ \frac{\partial^\alpha v(x, t)}{\partial t^\alpha} = d_v \frac{\partial^2 v(x, t)}{\partial x^2} - \beta v(x, t) + \gamma u(x, t) \end{cases} \quad (1)$$

with associated initial conditions

$$u(x, t) = u_0(x), \quad v(x, t) = v_0(x), \quad x \in I = (a, b) \quad (2)$$

where $0 < \alpha \leq 1$ is the fractional differential parameter, d_u , d_v , β and γ are various positive constants of biological importance (see Table 1 for their definitions and values), $u = u(x, t)$ and $v = v(x, t)$ are unknown state variables denoting the density of amoebae and concentration of chemoattractive substance, respectively and $\chi(v(x, t))$ represents the signal-dependent chemotactic sensitivity function. The chemotactic $\frac{\partial}{\partial x} \left(u \frac{\partial \chi(v(x, t))}{\partial x} \right)$ term appearing in the first equation of (1) measures sensitivity of the amoebae cells to the chemical substance. If, for instance, $\alpha = 1$ and $\chi(v) = \chi v$ with (*resp.* $\chi < 0$), the system (1) reduces to the classical one-dimensional attraction (*resp.* repulsion) Keller-Segel chemotaxis model (Keller *et al.*, 1970) which describes the aggregation dynamics of the amoeba *Dictyostelium discoideum* in response to cyclic Adenosine Monophosphate (cAMP) which mediate their aggregation. Generally, chemotaxis refers to the oriented motion of cellular species either in the direction of an attraction-type or away from a repulsion-type chemical signal. In biological processes, it accounts for cellular communication among motile marine organisms in their quest for mates, nutrients and survival. Among higher organisms, it dictates the processes of wound healing, pattern formation, cell-organization and positioning, embryogenesis, tumor cell invasion and cancer metastasis of living tissues. The classical Keller-Segel chemotaxis model (i.e., equation (1) with $\alpha = 1$) as well as several of its variant formulations have been extensively studied from different mathematical perspectives. For instance, it has been shown that the classical model admits globally bounded solutions in the one-dimensional settings (Hillen *et al.*, 2004; Yagi, 1997) whereas in higher dimensions a more complex dynamics arise in the sense that the solutions may blow up either in finite or infinite time (Blanchet *et al.*, 2006; Horstmann *et al.*, 2001; Senba *et al.*, 2001). Specifically, in the two-dimensional settings, it was conjectured that there exists a threshold value $M > 0$ for which the model admits global solution in time if $\int u_0(x) dx < M$ and for which blow up occurs if $\int u_0(x) dx > M$ (Childress *et al.*,

1984). A comprehensive review on the rich mathematical background concerning variant forms of the Keller-Segel chemotaxis model can be found in the survey articles (Hillin *et al.*, 2009; Murray, 2003; Perthame, 2007).

MATERIALS AND METHODS

Preliminaries

Definition 1. (Podlubny, 1999) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $\omega \in C_\mu (\mu \geq -1)$ is defined as

$$J_t^\alpha \omega(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\vartheta)^{\alpha-1} \omega(\vartheta) d\vartheta, & \alpha > 0, t > 0, \\ \omega(t), & \alpha = 0, t = 0, \end{cases} \quad (3)$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. (Caputo, 1969; Miler *et al.*, 1993; Kilbas *et al.*, 2006) The fractional derivative of order $\alpha > 0$ of a function $\omega \in C_{-1}^m$ in the sense of Caputo is defined as

$$D_t^\alpha \omega(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\vartheta)^{m-\alpha-1} \omega^{(m)}(\vartheta) d\vartheta, \quad (4)$$

where $m-1 < \alpha < m \in \mathbb{N}$. If $\alpha = m \in \mathbb{N}$, then

$$D_t^\alpha \omega(t) = \frac{d^m \omega(t)}{dt^m}.$$

Definition 3. (Caputo, 1969; Kilbas *et al.*, 2006) The Laplace transform of the Caputo fractional derivative $D_t^\alpha \omega(t)$ is defined as

$$\mathcal{L}[D_t^\alpha \omega(t)] = s^\alpha \mathcal{L}[\omega(t)](s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} \omega^{(k)}(0^+), \quad (5)$$

where $\alpha > 0$, $m-1 < \alpha < m$ and \mathcal{L} denotes the Laplace transform operator.

Basic solution procedure of the q -HATM

To demonstrate the solution procedure of the q -HATM, we consider the following general nonlinear time-fractional partial differential equation:

$$D_t^\alpha U(x, t) + \mathcal{R}U(x, t) + NU(x, t) = g(x, t), \quad (\alpha > 0, m-1 < \alpha < m, m \in \mathbb{N}) \quad (6)$$

where $D_t^\alpha U(x, t)$ represents the Caputo fractional derivative of an unknown function $U(x, t)$, \mathcal{R} is a bounded linear partial differential operator satisfying $\|\mathcal{R}U\| \leq \delta \|U\|$ for some $\delta > 0$, N is a nonlinear partial differential operator satisfying the Lipschitz condition: $\|\mathcal{R}U_1 - \mathcal{R}U_2\| \leq \varkappa \|U_1 - U_2\|$ for some $\varkappa > 0$ and $g(x, t)$ is a nonhomogeneous term. To initiate the q -HATM, we first take the Laplace transform of (6) and then use the differentiation property (5) to get

$$\mathcal{L}[U(x, t)] - \frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-k-1} U^{(k)}(x, 0) + \frac{1}{s^\alpha} (\mathcal{L}[\mathcal{R}U(x, t)] + \mathcal{L}[NU(x, t)] - \mathcal{L}[g(x, t)]) = 0 \quad (7)$$

after simplification. Next, we define the nonlinear operator

$$N[\phi(x, t; q)] = \mathcal{L}[\phi(x, t; q)] - \frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-k-1} \phi^{(k)}(x, t; q)(0^+) + \frac{1}{s^\alpha} (\mathcal{L}[\phi U(x, t)] + \mathcal{L}[\phi U(x, t)] - \mathcal{L}[g(x, t)]) \quad (8)$$

where $q \in [0, \frac{1}{n}]$, ($n \geq 1$) is the embedding parameter and $\phi(x, t; q)$ is an unknown real-valued function of x, t and q . For a nonzero auxiliary function $H(x, t)$, a homotopy is constructed in the following form:

$$(1-nq)\mathcal{L}[\phi(x, t; q) - U_0(x, t)] = \hbar q H(x, t) N\phi(x, t; q), \quad (9)$$

where \hbar is a nonzero auxiliary parameter and $U_0(x, t)$ is the initial assumption of U . Obviously, the following relation holds

$$\begin{cases} \phi(x, t; 0) = U_0(x, t), & \text{if } q = 0, \\ \phi(x, t; \frac{1}{n}) = U(x, t), & \text{if } q = \frac{1}{n}. \end{cases} \quad (10)$$

In other words, $\phi(x, t; q)$ varies from the initial guess $U_0(x, t)$ to the solution $U(x, t)$ as q varies from 0 to $\frac{1}{n}$. A Taylor's series expansion of $\phi(x, t; q)$ about q yields

$$\phi(x, t; q) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) q^m \quad (11)$$

where

$$U_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0} \quad (12)$$

For appropriate choices of the initial guess $U_0(x, t)$, the auxiliary parameter \hbar and the auxiliary function $H(x, t)$, the series (11) converges at $q = \frac{1}{n}$ and yields a solution

$$U(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) \left(\frac{1}{n}\right)^m \quad (13)$$

which is at least one solution of (6). Define the vectors

$$\vec{U}_m(x, t) = \{U_0(x, t), U_1(x, t), \dots, U_m(x, t)\}. \quad (14)$$

Taking the derivative of (9) m -times with respect to q , multiplying the result by $\frac{1}{m}$ and then setting $q = 0$, gives the m th-order deformation equation

$$\mathcal{L}[U_m(x, t) - k_m U_{m-1}(x, t)] = \hbar H(x, t) \mathfrak{R}_m(\vec{U}_{m-1}(x, t)) \quad (15)$$

where

$$\begin{aligned} \mathfrak{R}_m(\vec{U}_{m-1}) &= \mathcal{L}[U_{m-1}(x, t)] \\ &- \hbar \left(1 - \frac{k_m}{n}\right) \left(\frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-k-1} U^{(k)}(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[g(x, t)] \right) \\ &+ \frac{1}{s^\alpha} \mathcal{L}(\mathfrak{R}_m U_{m-1} + \mathcal{H}_{m-1}) \end{aligned} \quad (16)$$

and

$$k_m = \begin{cases} 0, & m \leq 1 \\ n, & m > 1 \end{cases} \quad (17)$$

In (16), \mathcal{H}_m denotes the homotopy polynomial which is defined as

$$\mathcal{H}_m = \frac{1}{m!} \left[\frac{\partial^m \phi(x, t; q)}{\partial q^m} \right]_{q=0} \text{ and } \phi(x, t; q) = \phi_0 + q\phi_1 + q^2\phi_2 + \dots \quad (18)$$

Applying the inverse transform to (3.10) yields the recursive equation

$$U_m(x, t) = k_m U_m(x, t) + \hbar \mathcal{L}^{-1}[H(x, t) \mathfrak{R}_m(\vec{U}_{m-1}(x, t))] \quad (19)$$

Thus, by substituting (16) into (19) we get

$$U_m(x, t) = (k_m + \hbar) \vec{U}_{m-1}(x, t) - \hbar \left(1 - \frac{k_m}{n} \right) \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-k-1} U^{(k)}(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[g(x, t)] \right] + \hbar \mathcal{L}^{-1} \left[\frac{1}{s^\alpha} \mathcal{L}(\mathfrak{R}_m U_{m-1} + \mathcal{H}_{m-1}) \right] \quad (20)$$

Finally, the approximate analytical solution of (6) is obtained by truncating the following series:

$$U_m(x, t) = U_0(x, t) + \sum_{m=1}^{\infty} U_m(x, t) \left(\frac{1}{n} \right)^m \quad (21)$$

The existence of the factor $\left(\frac{1}{n} \right)^m$ in the q -HATM solution (21) allows for faster convergence than the standard HAM. Moreover, in the special case $n = 1$, the q -HATM reduces to the standard homotopy analysis transform method (HATM).

q -HATM SOLUTION FOR TF-AKS MODEL

Here, we implement the q -HATM on the TF-AKS model (1) subject to the initial conditions (2). For the sake of simplicity, we assume a linear chemotactic sensitivity function in the sense that $\chi(v(x, t)) = v(x, t)$. To this end, we rewrite the system of equations (1) as

$$\begin{cases} N_1[\phi_1(x, t; q), \phi_2(x, t; q)] = \mathcal{L}[\phi_1(x, t; q)] - \frac{u_0(x, t)}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[-d_u \frac{\partial^2 \phi_1(x, t; q)}{\partial x^2} + \frac{\partial \phi_1(x, t; q)}{\partial x} \frac{\partial \phi_2(x, t; q)}{\partial x} + \phi_1(x, t; q) \frac{\partial^2 \phi_2(x, t; q)}{\partial x^2} \right] \\ N_2[\phi_1(x, t; q), \phi_2(x, t; q)] = \mathcal{L}[\phi_2(x, t; q)] - \frac{v_0(x, t)}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[-d_v \frac{\partial^2 \phi_2(x, t; q)}{\partial x^2} + \beta \phi_2(x, t; q) - \gamma \phi_1(x, t; q) \right] \end{cases}$$

The m th-order deformation equations with $H(x, t) = 1$ are constructed as

$$\begin{cases} \mathcal{L}[u_m(x, t) - k_m u_{m-1}(x, t)] = \hbar \mathfrak{R}_m^u[\vec{u}_{m-1}, \vec{v}_{m-1}], \\ \mathcal{L}[v_m(x, t) - k_m v_{m-1}(x, t)] = \hbar \mathfrak{R}_m^v[\vec{u}_{m-1}, \vec{v}_{m-1}], \end{cases} \quad (24)$$

where

$$\begin{cases} \mathfrak{R}_m^u[\vec{u}_{m-1}, \vec{v}_{m-1}] = \mathcal{L}[u_m(x, t)] - \left(1 - \frac{k_m}{n} \right) \frac{u_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[-d_u \frac{\partial^2 u_{m-1}}{\partial x^2} + \sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} \frac{\partial v_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} u_i \frac{\partial^2 v_{m-1-i}}{\partial x^2} \right], \\ \mathfrak{R}_m^v[\vec{u}_{m-1}, \vec{v}_{m-1}] = \mathcal{L}[v_m(x, t)] - \left(1 - \frac{k_m}{n} \right) \frac{v_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[-d_v \frac{\partial^2 v_{m-1-i}}{\partial x^2} + \beta v_{m-1-i} - \gamma u_{m-1-i} \right]. \end{cases} \quad (25)$$

Substituting (25) into (24) and then taking the inverse Laplace transform gives

Table 1: Model parameters description and values

Parameters	Description	Values	Reference
d_u	Diffusion coefficient of chemotactic cells	0.5	(Atangana, 2014)
d_v	Diffusion coefficient of chemoattractant	0.3	(Atangana, 2014)
γ	Rate of production of chemoattractant	1	(Atangana, 2014)
β	Rate of degradation of chemoattractant	2	(Atangana, 2014)

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} = d_u \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - u \frac{\partial^2 v}{\partial x^2}, \\ \frac{\partial^\alpha v}{\partial t^\alpha} = d_v \frac{\partial^2 v}{\partial x^2} - \beta v + \gamma u. \end{cases} \quad (22)$$

and apply the Laplace transform together with the differentiation property (5) to get

$$\begin{cases} \mathcal{L}[u(x, t)] - \frac{u_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[-d_u \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} \right] = 0 \\ \mathcal{L}[v(x, t)] - \frac{v_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[d_v \frac{\partial^2 v}{\partial x^2} + \beta v - \gamma u \right] = 0 \end{cases} \quad (23)$$

after simplification. Next, we define the following nonlinear operators:

$$\begin{cases} u_m = (k_m + \hbar)u_{m-1} - \hbar \left(1 - \frac{k_m}{n}\right) \mathcal{L}^{-1} \left\{ \frac{u_0}{s} \right\} \\ \quad + \hbar \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[-d_u \frac{\partial^2 u_{m-1}}{\partial x^2} + \sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} \frac{\partial v_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} u_i \frac{\partial^2 v_{m-1-i}}{\partial x^2} \right] \right\}, \\ v_m = (k_m + \hbar)v_{m-1} - \hbar \left(1 - \frac{k_m}{n}\right) \mathcal{L}^{-1} \left\{ \frac{v_0}{s} \right\} \\ \quad + \hbar \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[-d_v \frac{\partial^2 v_{m-1}}{\partial x^2} + \beta v_{m-1} - \gamma u_{m-1} \right] \right\} \end{cases} \quad (26)$$

from which all solution components of the TF-AKS model (22) can be generated. Moreover, the q -HATM series solution to the coupled system (22) is then given as

$$\begin{cases} u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m, \\ v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) \left(\frac{1}{n}\right)^m. \end{cases} \quad (27)$$

We refer to the Appendix for the convergence of the q -HATM series solution for the TF-AKS model. In what follows, we use the parameter values in Table 1 to obtain approximate analytical solutions for the TF-AKS model (22) for the three sets of initial data given in (28), (29) and (30). Numerical simulations demonstrating the biological behavior of the obtained q -HATM solutions are also provided in each case.

Case I

Consider the TF-AKS model (22) with the initial conditions $u(x, 0) = Me^{-x}$, $v(x, 0) = Ne^{-x}$. (28)

Thanks to the steps leading to (26), we have the following few solution iterations:

$$u_0(x, t) = Me^{-x},$$

$$v_0(x, t) = Ne^{-x},$$

$$u_1(x, t) = -\frac{\hbar M(-2e^{-2x} + e^{-x}d_u)t^\alpha}{\Gamma(\alpha + 1)},$$

$$v_1(x, t) = -\frac{\hbar e^{-x}(\gamma M + (-\beta + d_v)N)t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, t) = -\frac{(n+\hbar)\hbar M(-2e^{-2x} + e^{-x}d_u)t^\alpha}{\Gamma(\alpha + 1)} -$$

$$\frac{\hbar^2(-6N^2e^{-3x} - e^{-x}d_u^2 + 2(\gamma M - N\beta + 5Nd_u + Nd_v)e^{-2x})Mt^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$v_2(x, t) = -\frac{(n+\hbar)\hbar e^{-x}(\gamma M + (-\beta + d_v)N)t^\alpha}{\Gamma(\alpha + 1)}$$

$$-\frac{\hbar^2(2N\gamma Me^{-2x} + e^{-x}(M\beta\gamma - M\gamma d_u - M\gamma d_v - N\beta^2 + 2N\beta d_v - Nd_v^2))t^{2\alpha}}{\Gamma(2\alpha + 1)}.$$

Moreover, the remaining terms for $m \geq 3$ can be generated by following in the same procedure and the series solution is obtained according to (27)

Case II

Consider the TF-AKS model (22) with the initial conditions

$$u(x, 0) = Me^{-x^2}, \quad v(x, 0) = Ne^{-x^2}. \quad (29)$$

Then the steps leading to (26) yield the following few iterations:

$$u_0(x, t) = Me^{-x^2},$$

$$v_0(x, t) = Ne^{-x^2},$$

$$u_1(x, t)$$

$$= -\frac{2\hbar M(-e^{-2x^2}N(4x^2 - 1) + e^{-x^2}d_u(2x^2 - 1))t^\alpha}{\Gamma(\alpha + 1)},$$

$$v_1(x, t) = -\frac{\hbar e^{-x^2}(M\gamma + (4x^2d_v - \beta + 2d_v)N)t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, t)$$

$$= -\frac{2\hbar M(n + \hbar)(-e^{-2x^2}N(4x^2 - 1) + e^{-x^2}d_u(2x^2 - 1))t^\alpha}{\Gamma(\alpha + 1)}$$

$$-\frac{2M\hbar^2(-2N^2e^{-3x^2}(24x^4 - 18x^2 + 1) - 2e^{-x^2}d_u^2(4x^4 - 12x^2 + 3))t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$-\frac{2M\hbar^2((80Nx^4 - 116Nx^2 + 14N)d_u + (16Nx^4 - 36Nx^2 + 6N)d_v)e^{-2x^2}t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$-\frac{2M\hbar^2(4M\gamma x^2 - 4N\beta x^2 - M\gamma + N\beta)e^{-2x^2}t^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$v_2(x, t)$$

$$= -\frac{(n + \hbar)\hbar e^{-x^2}(\gamma M + (4x^2d_v - \beta - 2d_v)N)t^\alpha}{\Gamma(\alpha + 1)}$$

$$-\frac{2\hbar^2MN\gamma(4x^2 - 1)e^{-2x^2}t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$+\frac{\hbar^2((-16Nx^4 + 48Nx^2 - 12N)d_v^2 + (-4\gamma Mx^2 + 2\gamma M)d_u)e^{-2x^2}t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$-\frac{\hbar^2((-4\gamma Mx^2 + 8N\beta x^2 + 2\gamma M - 4N\beta)d_v + M\beta\gamma - N\beta^2)e^{-2x^2}t^{2\alpha}}{\Gamma(2\alpha + 1)}.$$

The remaining solution components for $m \geq 3$ can be generated by continuing in the same manner and the series solution with respect to the initial data (29) is obtained according to (27).

Case III

Consider the TF-AKS model (22) subject to the initial data:

$$u(x, 0) = M \sin x, \quad v(x, 0) = N \sin x. \quad (30)$$

From (26), we obtain the following few solution iterations:

$$u_0(x, t) = M \sin x,$$

$$v_0(x, t) = N \sin x,$$

$$u_1(x, t) = -\frac{M\hbar(-d_u \sin x + N(\sin^2 x - \cos^2 x))t^\alpha}{\Gamma(\alpha + 1)},$$

$$v_1(x, t) = -\frac{\hbar \sin x (M\gamma - N(\beta + d_v))t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, t) = -\frac{M\hbar(n + \hbar)(-d_u \sin x + N(-\cos^2 x + \sin^2 x))t^\alpha}{\Gamma(\alpha + 1)}$$

$$= -\frac{\hbar^2 M \left(-N^2 \sin^3 x + \cos^2 x (5N^2 \sin x + M\gamma - N(\beta + d_v + 5d_u)) \right) t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$- \frac{\hbar^2 M \left(((-M\gamma + N\beta + 5Nd_u + Nd_v) \sin x)^2 - d_u^2 \sin x \right) t^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$v_2(x, t) = -\frac{(n + \hbar)\hbar \sin x (M\gamma - N(\beta + d_v))t^\alpha}{\Gamma(\alpha + 1)}$$

$$+ \frac{\hbar^2 N \sin x (\beta + d_v)^2 t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$- \frac{\hbar^2 M \gamma (N(-\sin^2 x + \cos^2 x) + \sin x (\beta + d_v + d_u)) t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

The remaining solution iterates for $m \geq 3$ can be generated in the same manner and the series solution with respect to the initial data (30) is obtained according to (27).

RESULTS AND DISCUSSIONS

Here, we present numerical simulations for the TF-AKS model (22) with respect to the initial data (28), (29) and (30) with $M = 0.000012$ and $N = 0.000016$ using the parameter values provided in Table 1.

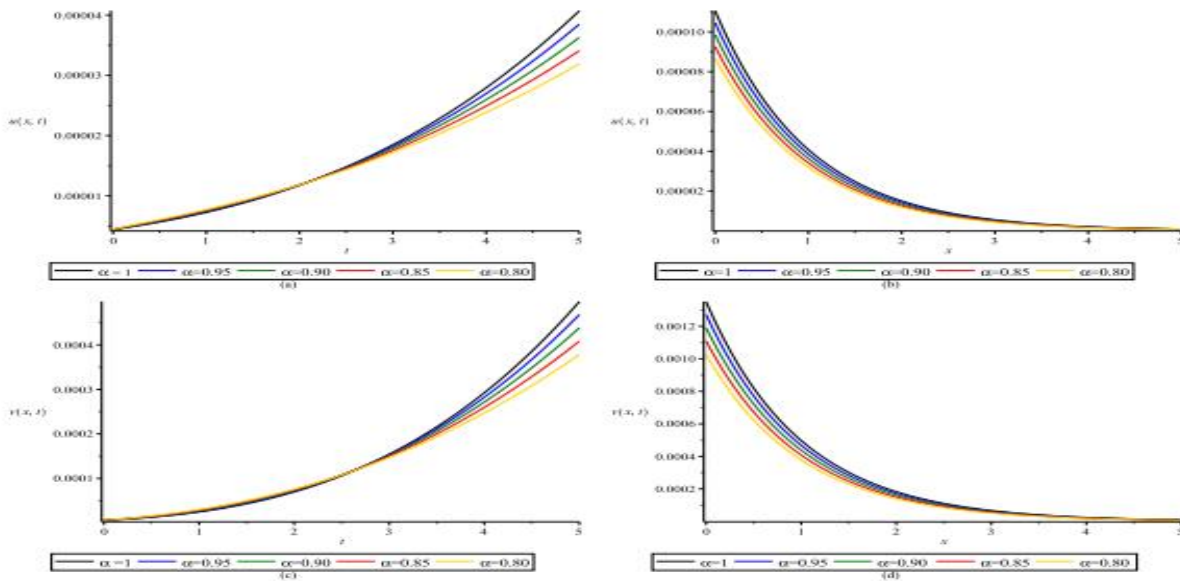


Figure 1: Numerical simulations for TF-AKS model (4.1) with initial data in CASE I, distinct values of α , $\hbar = -1$ and $n = 1$: (a) plot of amoeba density $u(x, t)$ vs. t at $x = 1$; (b) plot of amoeba density $u(x, t)$ vs. x at $t = 5$; (c) plot of chemoattractant concentration $v(x, t)$ vs. t at $x = 1$; (d) plot of chemoattractant concentration $v(x, t)$ vs. x at $t = 5$.

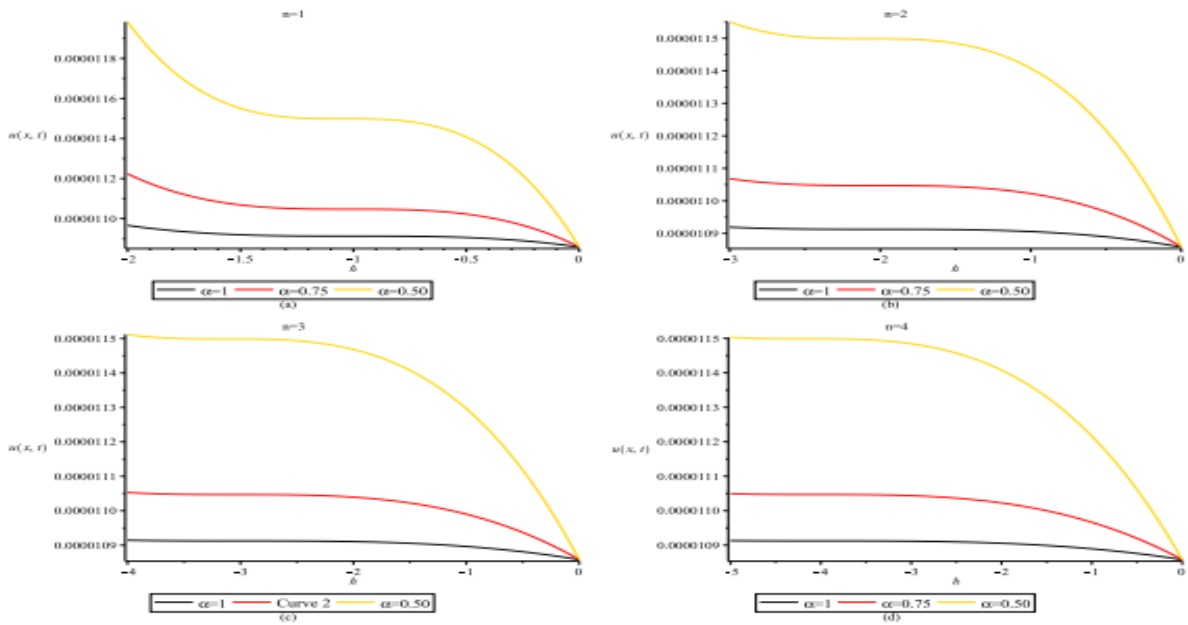


Figure 2: h -curves for $u(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE I: (a) at $n = 1$; (b) at $n = 2$; (c) at $n = 3$; (d) at $n = 4$

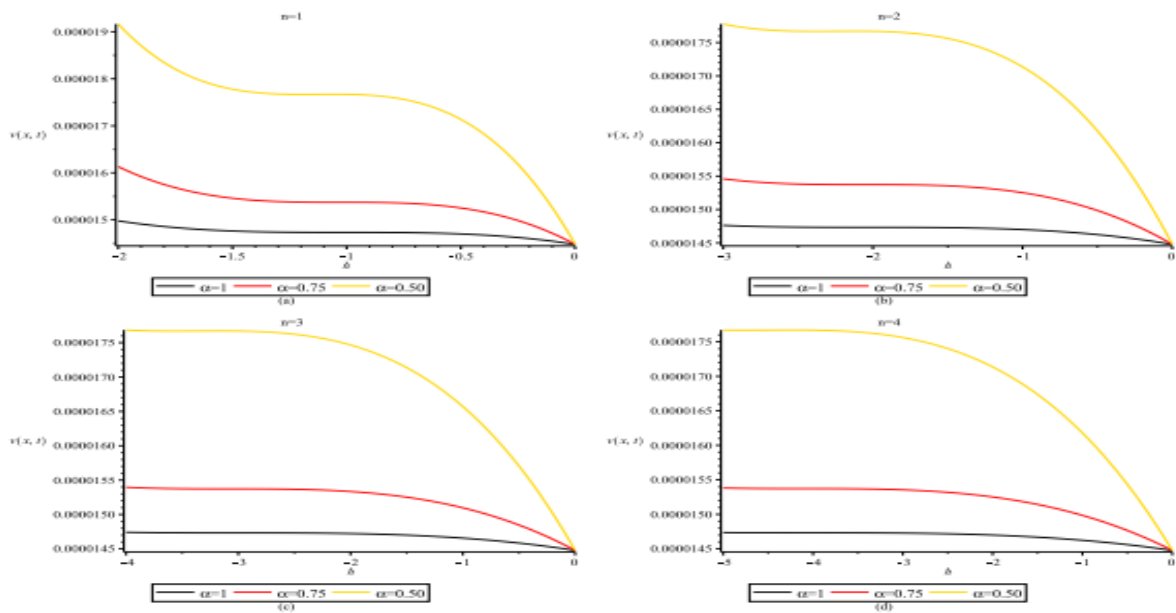


Figure 3: h -curves for $v(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE I: (a) at $n = 1$; (b) at $n = 2$; (c) at $n = 3$; (d) at $n = 4$.

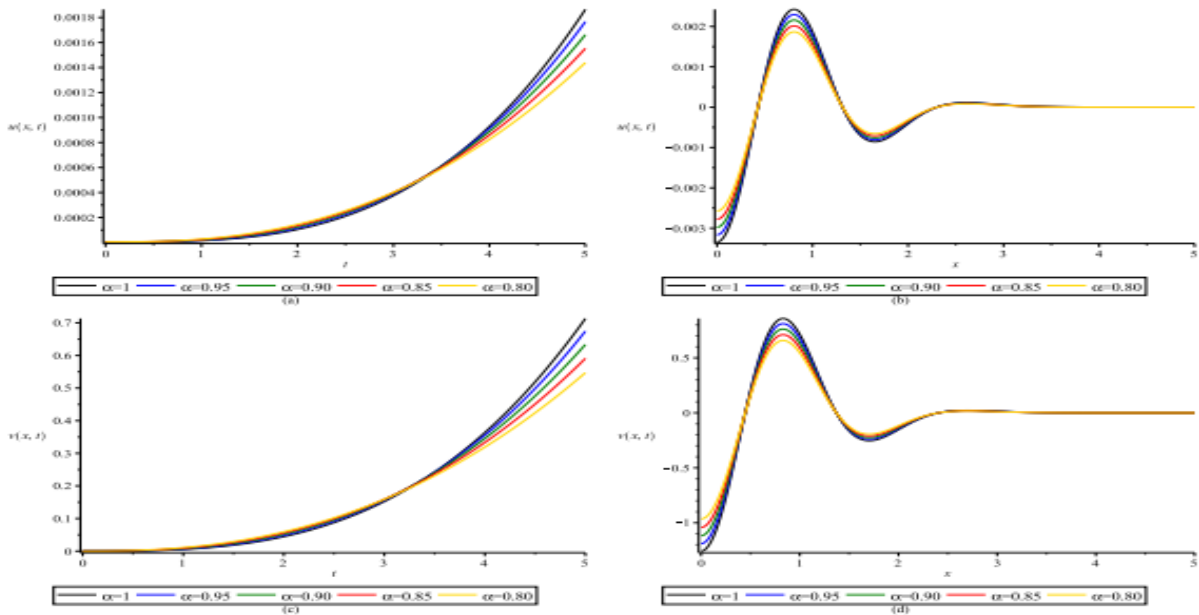


Figure 4: Numerical simulations for TF-AKS model (4.1) with initial data in CASE II, distinct values of α , $h = -1$ and $n = 1$: (a) plot of amoeba density $u(x, t)$ vs. t at $x = 1$; (b) plot of amoeba density $u(x, t)$ vs. x at $t = 5$; (c) plot of chemoattractant concentration $v(x, t)$ vs. t at $x = 1$; (d) plot of chemoattractant concentration $v(x, t)$ vs. x at $t = 5$.

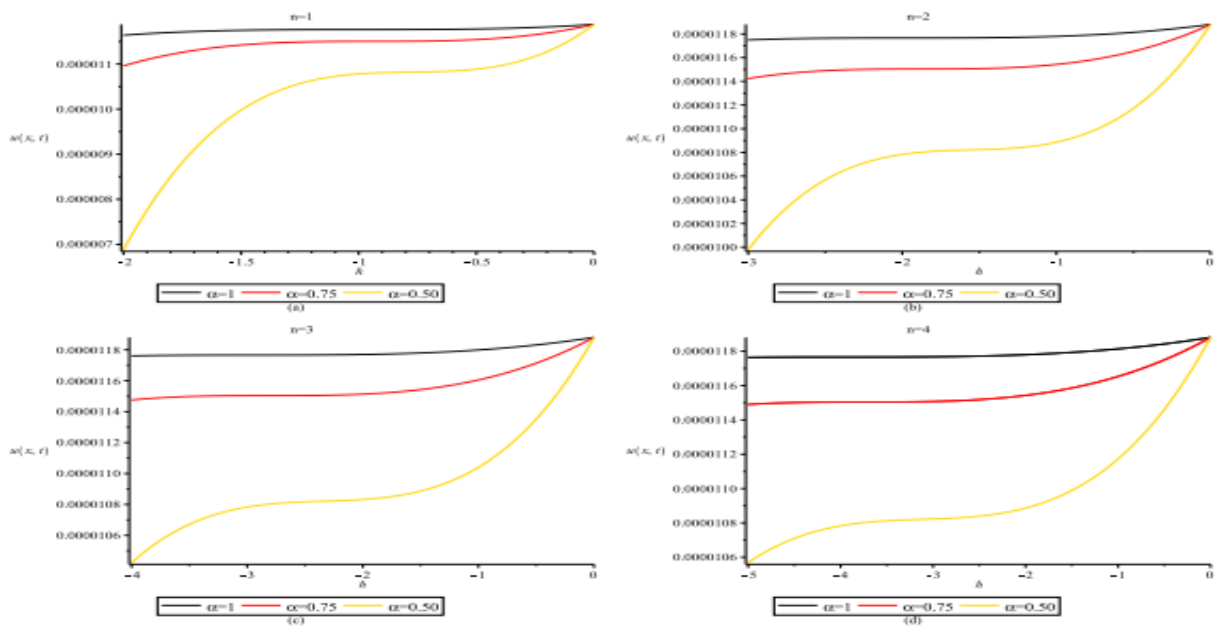


Figure 5: h -curves for $u(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE II: (a) at $n = 1$; (b) at $n = 2$; (c) at $n = 3$; (d) at $n = 4$.

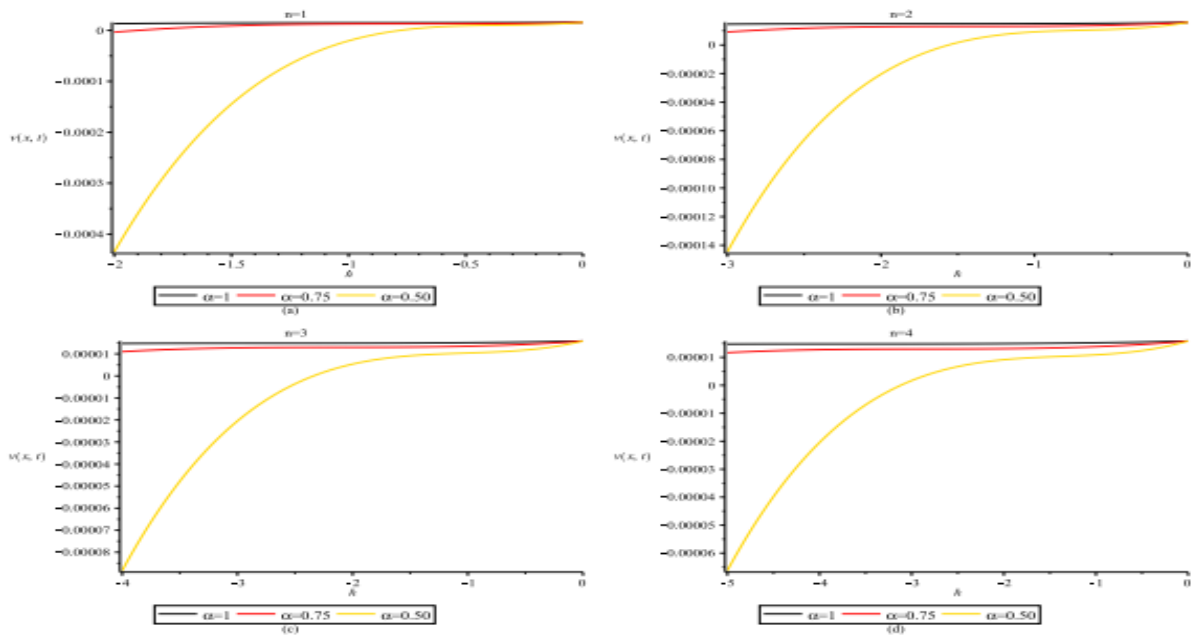


Figure 6: h -curves for $v(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE II: (a) at $n = 1$; (b) at $n = 2$; (c) at $n = 3$; (d) at $n = 4$.

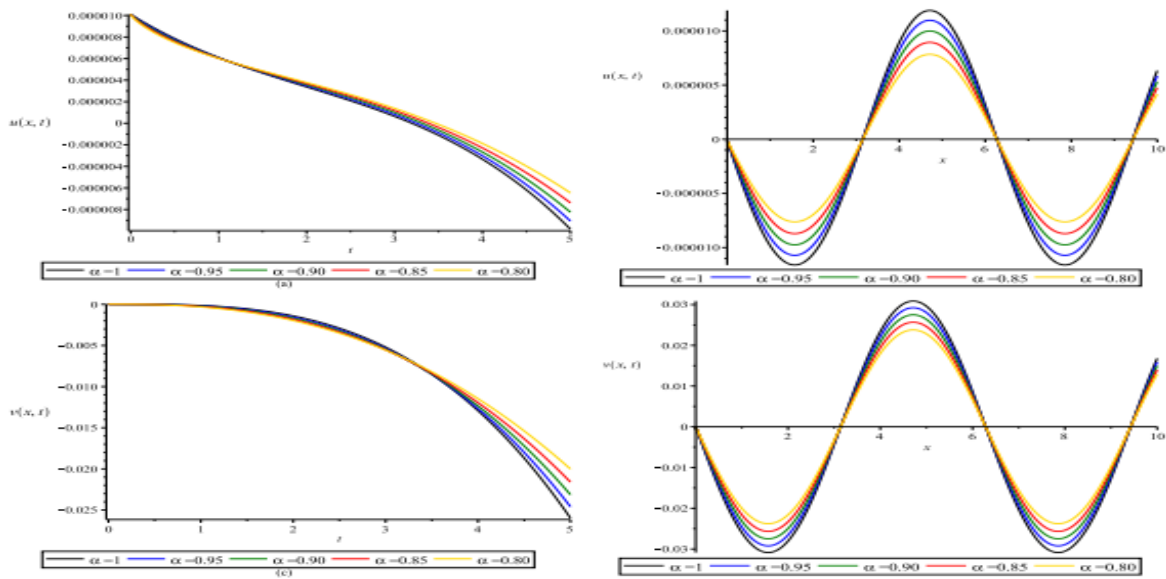


Figure 7: Numerical simulations for TF-AKS model (4.1) with initial data in CASE III, distinct values of α , $h = -1$ and $n = 1$: (a) plot of amoeba density $u(x, t)$ vs. t at $x = 1$; (b) plot of amoeba density $u(x, t)$ vs. x at $t = 5$; (c) plot of chemoattractant concentration $v(x, t)$ vs. t at $x = 1$; (d) plot of chemoattractant concentration $v(x, t)$ vs. x at $t = 5$.

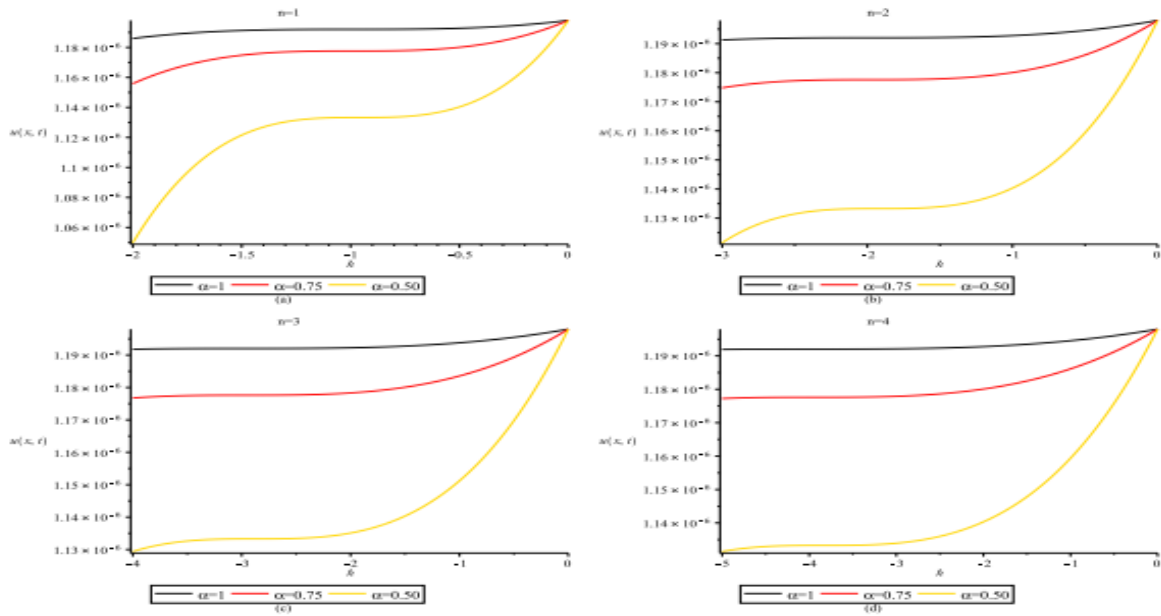


Figure 8: h -curves for $u(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE III: (a)at $n = 1$; (b)at $n = 2$; (c)at $n = 3$; (d)at $n = 4$.

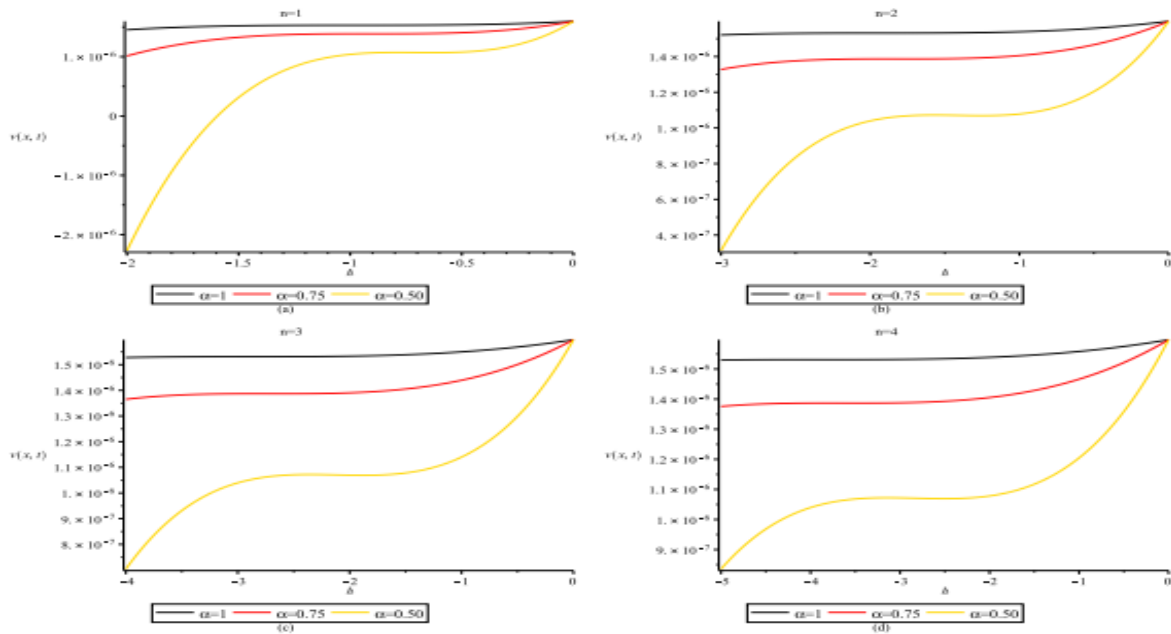


Figure 9: h -curves for $v(x, t)$ for distinct values of α when $x = 0.1$, $t = 0.01$ and initial value in CASE III: (a)at $n = 1$; (b)at $n = 2$; (c)at $n = 3$; (d)at $n = 4$.

The 2D-plots in Figure 1(a), Figure 4(a) and Figure 7(a) depict the behavior of the amoebae density $u(x, t)$ with respect to t for CASE I, CASE II and CASE III, respectively, at $x = 1$, $\hbar = -1$, $n = 1$ and distinct values of α . Figure 1(c), Figure 4(c) and Figure 7(c) depict the behavior of the concentration of the chemoattractant $v(x, t)$ with respect to t for CASE I, CASE II and CASE III, respectively, at $x = 1$, $\hbar = -1$, $n = 1$ and distinct values of α . In Figure 1(b), Figure 4(b) and Figure 7(b), show the behavior of the amoebae density $u(x, t)$ with respect to x for CASE I, CASE II and CASE III, respectively, at $t = 5$, $\hbar = -1$, $n = 1$ and distinct values of α . Figures 1(d), Figures 4(d) and Figures 7(d) capture the behavior of the concentration of the chemoattractant $v(x, t)$ with respect to x for CASE I, CASE II and CASE III, respectively, at $t = 5$, $\hbar = -1$, $n = 1$ and distinct values of α . In each case, whether for fixed t or fixed x , the plots demonstrate continuous dependence of the model solutions on the arbitrary fractional order α . Figure 2(a)-(d), Figure 5(a)-(d) and Figure 8(a)-(d) are \hbar -curves for the amoebae density $u(x, t)$ for initial data in CASE I, CASE II and CASE III, respectively, for distinct values of α when $x = 0.1$ and $t = 0.01$. Figure 3(a)-(d), Figure 5(a)-(d) and Figures 9(a)-(d) are \hbar -curves for the amoebae density $v(x, t)$ for initial data in CASE I, CASE II and CASE III, respectively, for distinct values of α when $x = 0.1$ and $t = 0.01$. For each case, whether for $u(x, t)$ or $v(x, t)$, the \hbar -curves are plotted for $n = 1$, $n = 2$, $n = 3$ and $n = 4$ as demonstrated in the figures. Furthermore, the horizontal line segments in these figures indicate the convergence range of the solution thus demonstrating the validity of the q -HATM solution in a very large domain.

Conclusion

In the present paper, approximate solutions for a one-dimensional TF-AKS chemotaxis model is investigated via the q -HATM for three sets of initial data. The time-fractional derivatives are taken in the sense of Caputo. The q -HATM which combines the classical q -HAM with parameter $q \in [0, \frac{1}{n}]$ ($n \geq 1$) and the usual Laplace transform method does not involve any form of linearization, discretization or restrictive assumption. The scheme also incorporates an auxiliary parameter \hbar which allows us to manipulate and control the series solution to ensure quick convergence. The behavior of the obtained series solution in comparison with varying fractional order parameter or auxiliary parameter \hbar are furnished via graphical representations. These graphs demonstrate continuous dependence of the model solutions on the fractional order parameter as well as chosen system parameters. In conclusion, we remark in view of the present work that the q -HATM is not only efficient and highly reliable but also a very effective analytical scheme in studying a wide class of coupled systems of nonlinear fractional differential equations describing a variety of biological phenomena as well as other systems arising in different fields of science and engineering.

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APPENDIX

Theorem A.1 (Uniqueness theorem) The solution to the TF-AKS model (4.1) determined by q -HATM is unique for every $0 \leq \lambda_1, \lambda_2 \leq 1$, where $\lambda_1 = (k_m + \hbar) - \hbar(d_u \delta^2 - \delta \gamma_1 - \gamma_2)T$ and $\lambda_2 = (k_m + \hbar) - \hbar(d_v \delta^2 - \beta)T$.

Proof. Let

$$\begin{cases} u(x, t) = \sum_{m=0}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m \\ v(x, t) = \sum_{m=0}^{\infty} v_m(x, t) \left(\frac{1}{n}\right)^m \end{cases} \quad (A.1)$$

be the q -HATM series solution to the TF-AKS model (22) where

$$\begin{cases} u_m = (k_m + \hbar)u_{m-1} - \hbar \left(1 - \frac{k_m}{n}\right) \mathcal{L}^{-1} \left\{ \frac{u_0}{s} \right\} \\ - \hbar \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[d_u \frac{\partial^2 u_{m-1}}{\partial x^2} - \sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} \frac{\partial v_{m-1-i}}{\partial x} - \sum_{i=0}^{m-1} u_i \frac{\partial^2 v_{m-1-i}}{\partial x^2} \right] \right\}, \\ v_m = (k_m + \hbar)v_{m-1} - \hbar \left(1 - \frac{k_m}{n}\right) \mathcal{L}^{-1} \left\{ \frac{v_0}{s} \right\} \\ - \hbar \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[d_v \frac{\partial^2 v_{m-1}}{\partial x^2} - \beta v_{m-1} + \lambda u_{m-1} \right] \right\} \end{cases} \quad (A.2)$$

Suppose the pairs (u, v) and (\tilde{u}, \tilde{v}) as distinct solutions of the TF-AKS model (22), then in view of the relation in (A.2), we get

$$\begin{aligned} |u - \tilde{u}| &\leq (k_m + \hbar)|u - \tilde{u}| - \hbar \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left(d_u \frac{\partial^2}{\partial x^2} |u - \tilde{u}| \right. \right. \\ &\quad \left. \left. - \frac{\partial}{\partial x} |u - \tilde{u}| \left| \frac{\partial v}{\partial x} \right| - |u - \tilde{u}| \left| \frac{\partial^2 v}{\partial x^2} \right| \right) \right\}. \end{aligned} \quad (A.3)$$

Furthermore, the convolution theorem for the Laplace transform ensures that we have

$$\begin{aligned} |u - \tilde{u}| &\leq (k_m + \hbar)|u - \tilde{u}| \\ &- \hbar \int_0^t \left(d_u \frac{\partial^2}{\partial x^2} |u - \tilde{u}| - \frac{\partial}{\partial x} |u - \tilde{u}| \left| \frac{\partial v}{\partial x} \right| \right. \\ &\quad \left. - |u - \tilde{u}| \left| \frac{\partial^2 v}{\partial x^2} \right| \right) \frac{(t - \vartheta)^\alpha}{\Gamma(\alpha + 1)} d\vartheta \\ &\leq (k_m + \hbar)|u - \tilde{u}| \\ &- \hbar \int_0^t (d_u \delta^2 |u - \tilde{u}| - \delta |u - \tilde{u}| \gamma_1 \\ &\quad - |u - \tilde{u}| \gamma_2) \frac{(t - \vartheta)^\alpha}{\Gamma(\alpha + 1)} d\vartheta, \end{aligned}$$

where $\delta^2 = \frac{\partial^2}{\partial x^2}$, $\delta = \frac{\partial}{\partial x}$, $\left| \frac{\partial v}{\partial x} \right| \leq \gamma_1$ and $\left| \frac{\partial^2 v}{\partial x^2} \right| \leq \gamma_2$.

Moreover, by the integral mean value theorem, we have

$$\begin{aligned} |u - \tilde{u}| &\leq (k_m + \hbar)|u - \tilde{u}| \\ &- \hbar(d_u \delta^2 |u - \tilde{u}| - \delta |u - \tilde{u}| \gamma_1 \\ &\quad - |u - \tilde{u}| \gamma_2) T \\ &= [(k_m + \hbar) - \hbar(d_u \delta^2 - \delta \gamma_1 - \gamma_2)T] |u - \tilde{u}|, \end{aligned}$$

that is $(1 - \lambda_1)|u - \tilde{u}| \leq 0$ where $\lambda_1 = (k_m + \hbar) - \hbar(d_u \delta^2 - \delta \gamma_1 - \gamma_2)T$. Since $0 < \lambda_1 < 1$, then $|u - \tilde{u}| = 0$. Thus $u = \tilde{u}$. In a similar manner, we also get $v = \tilde{v}$ where $\lambda_2 = (k_m + \hbar) - \hbar(d_v \delta^2 - \beta)T$. This completes the prove of uniqueness of the solution.

Theorem A.2 (Convergence theorem) Let χ a Banach space and $\Pi: \chi \rightarrow \chi$ a nonlinear mapping? Assume that

$$\|\Pi(\varpi) - \Pi(\omega)\| \leq \eta \|\varpi - \omega\|, \quad \forall \varpi, \omega \in \chi,$$

then Π admits a fixed point in view of the Banach fixed point theorem [4]. Moreover, for any $\varpi_0, \omega_0 \in \chi$, the series generated by the q -HATM converges to a fixed point of Π and

$$\|\varpi_m - \varpi_n\| \leq \frac{\eta^n}{1 - \eta} \|\varpi_1 - \varpi_0\|, \quad \forall \varpi, \omega \in \chi.$$

Proof. Let $\varpi := \{u, v\}$ and $\eta := \{\lambda_1, \lambda_2\}$ where u is the amoebae density and v is the concentration of the chemoattractant. Assume that $(C[J], \|\cdot\|)$ is a Banach space of all continuous functions on J with norm: $\|w(t)\| := \max_{t \in J} |w(t)|$, then by first taking $\varpi := u$ and $\eta := \lambda_1$, we prove that $\{u_i\}$ is a Cauchy sequence in the Banach space. To this end, consider

$$\begin{aligned} \|u_m - u_i\| &= \max_{t \in J} |u_m - u_i| \\ &= \max_{t \in J} \left| (k_m + \hbar)(u_{m-1} - u_{i-1}) \right. \\ &\quad - \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[d_u \left(\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial^2 u_{i-1}}{\partial x^2} \right) \right. \right. \\ &\quad \left. \left. - \left(\frac{\partial u_{m-1}}{\partial x} - \frac{\partial u_{i-1}}{\partial x} \right) \left| \frac{\partial v}{\partial x} \right| \right. \right. \\ &\quad \left. \left. - (u_{m-1} - u_{i-1}) \left| \frac{\partial^2 v}{\partial x^2} \right| \right] \right\} \\ &\leq \max_{t \in J} (k_m + \hbar) |u_{m-1} - u_{i-1}| \\ &\quad - \mathcal{L}^{-1} \left\{ \frac{1}{s^\alpha} \mathcal{L} \left[d_u \left| \frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial^2 u_{i-1}}{\partial x^2} \right| \right. \right. \\ &\quad \left. \left. - \left| \frac{\partial u_{m-1}}{\partial x} - \frac{\partial u_{i-1}}{\partial x} \right| \left| \frac{\partial v}{\partial x} \right| \right. \right. \\ &\quad \left. \left. - |u_{m-1} - u_{i-1}| \left| \frac{\partial^2 v}{\partial x^2} \right| \right] \right\}. \end{aligned}$$

By the convolution theorem for Laplace transform, we get

$$\begin{aligned} \|u_m - u_i\| &\leq \max_{t \in J} \left[(k_m + \hbar) |u_{m-1} - u_{i-1}| \right. \\ &\quad - \hbar \int_0^t \left(d_u \left| \frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial^2 u_{i-1}}{\partial x^2} \right| \right. \\ &\quad \left. - \left| \frac{\partial u_{m-1}}{\partial x} - \frac{\partial u_{i-1}}{\partial x} \right| \left| \frac{\partial v}{\partial x} \right| \right. \\ &\quad \left. - |u_{m-1} - u_{i-1}| \left| \frac{\partial^2 v}{\partial x^2} \right| \right) \frac{(t - \vartheta)^\alpha}{\Gamma(\alpha + 1)} d\vartheta \\ &\leq \max_{t \in J} \left[(k_m + \hbar) |u_{m-1} - u_{i-1}| \right. \\ &\quad - \hbar \int_0^t (d_u \delta^2 |u_{m-1} - u_{i-1}| \\ &\quad - \delta |u_{m-1} - u_{i-1}| \gamma_1 \\ &\quad \left. - |u_{m-1} - u_{i-1}| \gamma_2) \frac{(t - \vartheta)^\alpha}{\Gamma(\alpha + 1)} d\vartheta \right] \end{aligned}$$

By the integral mean value theorem, the relation above reduces to

$$\begin{aligned} \|u_m - u_i\| &\leq \max_{t \in J} [(k_m - \hbar) |u_{m-1} - u_{i-1}| \\ &\quad - \hbar (d_u \delta^2 |u_{m-1} - u_{i-1}| \\ &\quad - \delta |u_{m-1} - u_{i-1}| \gamma_1 \\ &\quad - |u_{m-1} - u_{i-1}| \gamma_2) T]. \end{aligned}$$

That is, $\|u_m - u_i\| \leq \lambda_1 \|u_{m-1} - u_{i-1}\|$. Since $0 < \lambda_1 < 1$, so $1 - \lambda_1^{m-i-1} < 1$, we have

$$\|u_m - u_i\| \leq \frac{\lambda_1^i}{1 - \lambda_1} \|u_1 - u_0\|.$$

But $\|u_1 - u_0\| < \infty$, consequently as $m, i \rightarrow \infty$ than $\|u_1 - u_0\| \rightarrow 0$ and therefore, the sequence $\{u_i\}$ is Cauchy in $C[J]$. A similar line of reasoning yield that the sequence $\{v_i\}$ is also Cauchy sequence in $C[J]$. Hence, $\{u_i\}$ and $\{v_i\}$ are convergent sequences. This concludes the proof of the theorem.

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