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SOLUTION OF SYSTEMS OF DISJOINT FREDHOLM-VOLTERRA INTEGRO- DIFFERENTIAL EQUATIONS USING BEZIER CONTROL POINTS

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ABSTRACT

Systems of disjoint Fredholm-Volterra integro-differential equations and the Bezier curves control-point-based algorithm are considered. Systems of two, three and four Fredholm-Volterra integro-differential equations are solved using a developed algorithm. The convergence analysis for the Bezier curves method proves that it is convergent. The examples considered agree with the convergence analysis. The method is more accurate and effective when compared to other existing methods.

Keywords: Bezier Curves, Control Points, Fredholm-Volterra Integro-Differential Equations, Residual Function, Convergence Analysis

MSC2010: 45J99, 65D15.

INTRODUCTION

Fredholm-Volterra integro-differential equations are used in modelling several problems in science and engineering. To this end, several methods have been proposed in literature to solve this class of equations. For instance, Arqub *et al.* (2009) used the method of reproducing kernel to solve Fredholm-Volterra integro-differential equations. The advantage of this method is that it is possible to pick any point in the interval of integration and as well the approximate solution and its derivative will be applicable. Also, Berenguer *et al*. (2012) introduced a solution method based on the fixed point iterative algorithm while Maleknejad *et al.* (2012) presented a Bernstein operational-matrix-based method for the solution of Fredholm-Volterra integro-differential equations. In Maleknejad and Attary (2012), a Chebyshev collocation points together with the Shannon approximation was introduced to transform Fredholm-Volterra integro-differential equations system to an algebraic system. Ibrahim and Ayoo (2015) used a new iterative method to solve integro-differential equations which yielded accurate results. These methods usually transform the Fredholm-Volterra integro-differential equations into a system of equations that can be solved by direct or iterative methods.

The use of Bezier curves method in numerical study of integral equations has also proven successful in some works. This technique was used by Ghomanjani *et al*. (2013a; 2013b) to solve Volterra type linear integro-differential equations systems and fourth order integro-differential equations respectively. Ayoo *et al.* (2016) solved linear and nonlinear Fredholm-Volterra integral equations using this same technique while Baydas & Karakas (2019) presented a way of finding points which presents a curve with a coordinate function as a Bezier curve.

The novel idea in this work however, is to use the Bezier curves technique to solve systems of disjoint Fredholm-Volterra integro-differential equations.

MATERIALS AND METHODS

Algorithm for the Bezier Control Points Method

Let us consider the following disjoint Fredhom-Volterra integro-differential equation

$$y^{(k)}(t) = x(t) + \tau_1 \int_{t_0}^{t_f} k_1(t,s,y(s))ds + \tau_2 \int_{t_0}^{t_f} k_2(t,s,y(s))ds, \quad t\epsilon[t_0,t_f] \tag{1}$$

with initial conditions $y^{(0)}(t_0) = a_0, y^{(1)}(t_0) = a_1, \dots, y^{(k-1)}(t_0) = a_{k-1}$.

We desire to approximate the solution $y(t)$ using the Bezier curves. We choose the sum of squares or the Euclidean norm of the Bezier control points of the residual to be the measure quantity. Minimizing this quantity gives the approximate solution. If the minimization of this quantity is zero, then the residual function is also zero, implying that the solution is the exact solution.

We present the algorithm (Ghomanjani & Farahi 2012) of this method for the Fredholm-Volterra integro-differential equations. The algorithm for this Bezier control-point-based approach is as follows:

Step 1:

Choose a degree of n and symbolically express the solution $y(t)$ in the degree n $(n \geq m)$ Bezier form

$$y(t) = \sum_{r=0}^{n} a_r B_{r,n}\left(\frac{t-t_0}{h}\right) \tag{2}$$

where $h = t_f - t_0$ and

$$B_{r,n}\left(\frac{t-t_0}{h}\right) = \binom{n}{r}\frac{1}{h^n}(t_f - t)^{n-r}(t-t_0)^r \tag{3}$$

and the control points $a_0, a_1, \dots, a_n$ are to be determined.

Step 2:

Substitute the approximate solution $y = y(t)$ into equations (1) to obtain the residual function

$$R(t) = y^{(k)}(t) - \left(x(t) + \tau_1 \int_{t_0}^{t_f} k_1(t,s,y(s))ds + \tau_2 \int_{t_0}^{t_f} k_2(t,s,y(s))ds\right)$$

This is a polynomial in t with degree $\leq k$.

So the residual function $R(t)$ can be expressed in the Bezier form as

$$R(t) = \sum_{r=0}^{n} b_r B_{r,k}(t) \quad (4)$$

where the control points $b_0, b_1, \dots, b_k$ are linear functions in the unknowns a_r. We derive these functions using the operations of multiplication, degree elevation and differentiation for Bezier form.

Step 3:

Construct the objective function

$$F = \sum_{r=0}^{k} b_r^2 = \int_{t_0}^{t_f} ||R(t)||^2 dt$$

where $||.||$ is the Euclidean norm. Then F is also a function of $a_0, a_1, \dots, a_k$.

Step 4:

Solve the constrained optimization problem

$$\text{Minimize } F \quad (5)$$

$$\frac{d^r y(0)}{dt^r} = \alpha_r, \frac{d^r y(1)}{dt^r} = \beta_r, r = 0,1,\dots,m-1$$

Step 5:

Substitute the minimum solution back into (2) to arrive at the approximate solution to the integro-differential equation.

Convergence of the Method

Using the concept of uniform convergence and degree elevation for Bezier functions (Ayoo *et al.* 2016), we consider the following problem based on disjoint Fredholm-Volterra integro-differential equation.

$$L_1(t,y(t)) = y^k(t) - \int_0^1 k_1(t,s,y(s))ds - \int_0^t k_2(t,s,y(s))ds = x(t), \quad t\epsilon[0,1] \quad (6)$$

$$y^{(0)}(t_0) = a_0, y^{(1)}(t_0) = a_{1,\dots,} y^{(k-1)}(t_0) = a_{k-1.}$$

where a is given real number and $k_{1,2}(t,s)\, \epsilon\, L^2[0,1]$ and $x(t)\, \epsilon\, L^2[0,1]$ are known functions for $t\, \epsilon\, [t_0, t_f]$, in particular [0,1]. Convergence of the approximate solution is done in degree rising of the Bezier polynomial approximation.

Theorem 3.1 (Ghomanjani & Farahi 2012)

If the integro-differential equation (6) has a unique C^1 continuous solution $\overline{y}$, then the approximate solution y obtained by the control-point-based method converges to the exact solution $\bar{y}$ as the degree of the approximate solution tends to infinity.

Proof

For arbitrary small positive number $\epsilon > 0$, by the Weiestrass theorem, one can find a polynomial $Q_{1,N_1}(t)$ of degree N such that

$$\|Q_{1,N_1}(t) - \bar{y}(t)\|_\infty \leq \frac{\in}{16}$$

where $\|\,.\,\|$ stands for the L_∞– norm over [0,1]. In particular, we have

$$\|a - Q_{1,N_1}(0)\| - \infty \leq \frac{\epsilon}{16}$$

Generally, $Q_{1,N_1}(t)$ does not satisfy the boundary conditions. With a small perturbation with a constant polynomial a, for $P_{1,N_1}(t)$, we can get the polynomial $P_{1,N_1}(t) = Q_{1,N_1}(t) + a$ such that $P_{1,N_1}(t)$ satisfy the boundary condition $P_{1,N_1}(0) = a$. Thus $F_{1,N_1}(0) + a = a \Rightarrow Q_{1,N_1}(0) = a - a$. From equation (6),

$$\|a - Q_{1,N_1}(0)\|_\infty = \|a - (a + a)\|_\infty = \|a - a + a\|_\infty$$

$$\|a - Q_{1,N_1}(0)\| = \|a\|_\infty \leq \frac{\epsilon}{16}$$

$$\|P_{1,N_1} - y(t)\|_\infty = \|Q_{1,N_1}(t) + a - y(t)\|_\infty$$

$$\leq \|Q_{1,N_1}(t) - y(t)\|_\infty + \|a\|_\infty$$

$$\leq \frac{\epsilon}{16} + \frac{\epsilon}{16} = \frac{\epsilon}{8} < \frac{\epsilon}{6}.$$

Let

$$LP_N(t) = L\left(t, P_{1,N_1}(t)\right)$$

$$= P_{1,N_1}(t) - \int_0^1 k_1(t,s,y(s))ds - \int_0^t k_2(t,s,y(s))ds = x(t)$$

For every $t \in [\,0,1]$. For $N \geq N_1$, the upper bound of the residual may be found

$$\|LP_N(t) - y(t)\|_\infty = \|L(t, P_{1,N_1}(t) - y(t)\|_\infty$$

$$\geq \|P_{1,N_1}(t) - y(t)\| + \int_0^1 \left\|k_1\left(t,s,P_{1,N_1}(s)\right)ds\right\|_\infty + \int_0^t \left\|k_2\left(t,s,P_{1,N_1}(s)\right)ds\right\|_\infty$$

$$\leq c_1\left(\frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6}\right) = c_1\frac{\epsilon}{2}$$

$$\leq c_1\epsilon$$

Where $c_1 = 1 + \|k_1(t,s)\|_\infty + \|k_2(t,s)\|_\infty$ is a constant.

The residual $R(P_N) = LP_N(t) - y(t)$ is considered as a polynomial; if not so, we can make use of the Taylor series to express it. Representing the residual $R(P_N)$ in Bezier form, we have

$$R(P_N) = \sum_{r=0}^{m_1} d_{r,m_1} B_{r,m_1}(t) \quad (7)$$

There exists an integer $M \geq N$ such that

$$\left|\frac{1}{m+1}\sum_{r=0}^{m_1} d_{r,m_1}^2 - \int_0^1 \left(R(P_N)\right)^2 dt\right| < \epsilon$$

$$\frac{1}{m+1}\sum_{r=0}^{m_1} d_{r,m_1}^2 < \epsilon + \int_0^1 \left(R(P_N)\right)^2 dt \leq \epsilon + (c_1\epsilon) \quad (8)$$

If $y(t)$ is an approximate solution of (6) gotten from the Bezier curves method of $m_2 (m_2 \geq m_1 \geq M)$.

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Let
$R(t,y(t)) = L(t,y(t)-y(t))$
$= \sum_{r=0}^{m_2} c_r, m_{2,} B_{r,m_2}(t),\ m_2 \geq m_1 \geq M$
The norm for difference-approximated solution $y(t)$ and exact solution $\bar{y}(t)$ is
$$\|y(t)-\bar{y}(t)\| = \int_0^1 \|y(t)-\bar{y}(t)\| dt \quad (9)$$
It can be shown that
$$\|y(t)-\bar{y}(t)\| \leq c\Big(|y(0)-\bar{y}(0)| + \big\|R(t,y(t)) - R(t,\bar{y}(t))\big\|_2^2\Big)$$
$$= c\int_0^1 \sum_{r=0}^{m_2} c_r, m_{2,} B_{r,m_2}(t)\, dt$$
$$\leq \frac{c}{m_2+1}\sum_{r=0}^{m_2} c_r^2, m_2$$
This inequality is arrived at using by uniform convergence (Ayoo *et al.* 2016), where c is a constant positive number. Thus by (7)
$$\|y(t)-\bar{y}(t)\| \leq \frac{c}{m_2+1}\sum_{r=0}^{m_2} c_r^2, m_2$$
$$\leq \frac{c}{m_2+1}\sum_{r=0}^{m_2} d_r^2, m_2$$
$$\leq \frac{c}{m_1+1}\sum_{r=0}^{m_2} d_r^2, m_1$$
$\leq c(\epsilon + c_1^2\epsilon^2) = \epsilon, m_1 \geq$ M
This inequality is arrived at from (8). Thus
$$\|y(t)-\bar{y}(t)\| \leq \epsilon_1$$
The infinite norm and the norm in (9) are equivalent, therefore, there exist a $\rho_1 > 0$ such that
$$\|y(t)-\bar{y}(t)\|_\infty \leq \rho_1\epsilon_1 = \epsilon_2.$$

RESULTS AND DISCUSSION

Results

We now present some numerical results to attest to the effectiveness of the method.

Example 1: Consider the following disjoint Fredholm-Volterra integro-differential equation (Wazwaz 2011)
$$y'(t) = 11 + 17t - 2t^3 - 3t^4 + \int_0^t su(s)ds + \int_0^1 (t-s)u(s)ds$$
with the initial condition $y(0) = 0$ and the exact solution $y(t) = 6t + 12t^2$.

The residual function is
$$R(t) = y'(t) - 11 - 17t + 2t^3 + 3t^4 - \int_0^t sy(s)ds - \int_0^1 (t-s)y(s)ds$$

When $n = 8$ the control points are
$a_0 = -6.031539495 \times 10^{307}$,
$a_1 = 0.74999999999992560$,
$a_2 = 1.92857142857142838$,
$a_3 = 3.53571428571426782$,
$a_4 = 5.57142857142854542$,
$a_5 = 8.03571428571428292$,
$a_6 = 10.9285714285714236$,
$a_7 = 14.2500000000000000$,
$a_8 = 18.0$.

When we substitue the control points into (2), we obtain the approximate solution
$$u(t) = -6.031539495 \times 10^{-307} + 6.0t + 12.00000001t^2 - 0.0000001t^3 + 0.0000001t^4 + 4.825231596 \times 10^{-306}t^7$$

Table I: Comparison of Absolute errors for example 1 ($n = 8$)

t	Fixed Point Iterative	Bernstein Operational Matrix	Bezier Control Points
0	0.1000	1.1006×10^{-2}	0.0
0.1	2.5660×10^{-9}	2.0110×10^{-10}	1.0000×10^{-11}
0.2	2.4877×10^{-10}	7.4579×10^{-9}	2.4000×10^{-10}
0.3	8.5671×10^{-9}	8.9620×10^{-9}	9.9000×10^{-10}
0.4	3.5712×10^{-7}	4.2400×10^{-9}	2.2400×10^{-9}
0.5	3.6667×10^{-9}	8.6500×10^{-9}	3.7500×10^{-9}
0.6	5.0045×10^{-8}	4.1400×10^{-8}	5.0400×10^{-9}
0.7	7.8900×10^{-7}	5.8952×10^{-8}	5.3900×10^{-9}
0.8	4.8720×10^{-6}	5.8920×10^{-9}	3.8000×10^{-9}
0.9	7.1570×10^{-7}	8.1570×10^{-9}	8.1000×10^{-10}
1.0	2.3000×10^{-8}	1.0530×10^{-7}	1.0000×10^{-8}

Example 2: Consider the following systems of disjoint Fredholm-Volterra integro-differential equations, with the exact solutions; $y_1(t) = 1 + 28t, y_2(t) = 1 - 6t - 2t^2$ respectively.
$$y''_1(t) = 6 - 6t - \frac{1}{2}t^2 - t^3 + \int_0^t (t-s)y_1(s)ds + \int_{-1}^1 ty_1(s)ds$$
$$y''_2(t) = -8 - \frac{1}{2}t^2 + t^3 + \frac{3}{4}t^4 + \int_0^t sy_2(s)ds + \int_{-1}^1 (t-s)y_2(s)ds$$
For the first equation, we have the approximate solution;
$$u_1(t) = 1.0 + 26.82018247t + 2.9999925t^2 - 1.9995924t^3 - 0.000493t^4 + 0.174610t^5 + 0.0071029t^6 - 0.0018028t^7$$

And for the second equation, we have;
$$u_2(t) = 1.0 - 6.083276801t - 1.987256524t^2 + 0.11291998t^3 + 0.00059364t^4 - 0.0527463t^5 + 0.0099865t^6 - 0.00022037t^7$$

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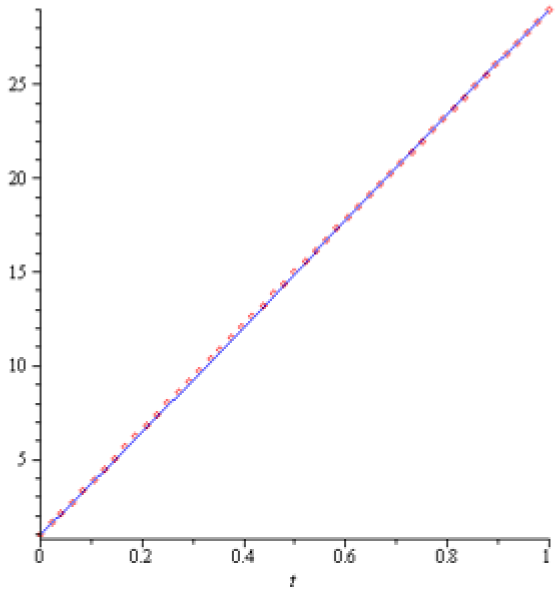
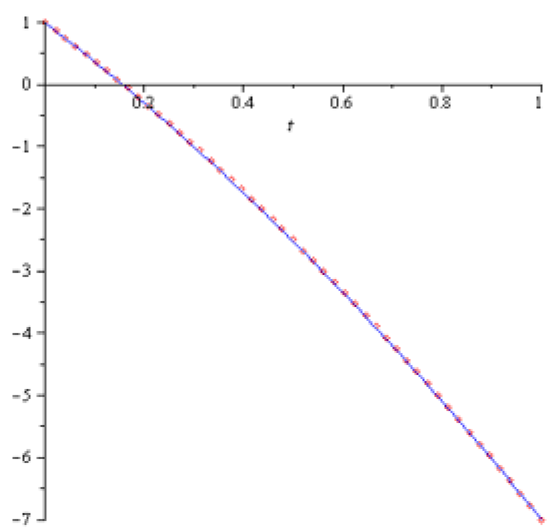


1 (a)



1(b)

Figure 1a-b: Exact and approximate solutions of Example 2 [▫▫▫ Exact --- Bezier]

Example 3: Consider the following systems of disjoint Fredholm-Volterra integro-differential equations, with the exact solutions; $y_1(t) = 65t + 10t^3, y_2(t) = \frac{1}{30} + 25t, y_3(t) = 6t$ respectively.

$$y'_1(t) = -6 - 2t + 19t^3 - t^5 + \int_0^t (t-s)y_1(s)ds + \int_0^1 (t+s)y_1(s)ds$$

$$y'_2(t) = -1 - 3t^2 - 2t^3 + \int_0^t (t-s)y_2(s)ds + \int_0^1 (t+s)y_2(s)ds$$

$$y'_3(t) = 4 - t - 4t^2 - t^3 + \int_0^t (t-s+1)y_3(s)ds + \int_0^1 (t+s-1)y_3(s)ds$$

For the first equation, we have;

$$u_1(t) = -3.14719289450058300 \times 10^{-306} + 64.37637463t + 4.6750291t^2 - 0.950619t^3 + 6.930721t^4 + 0.107218t^5 - 0.170624t^6 + 0.031898t^7$$

For the first equation, we have;

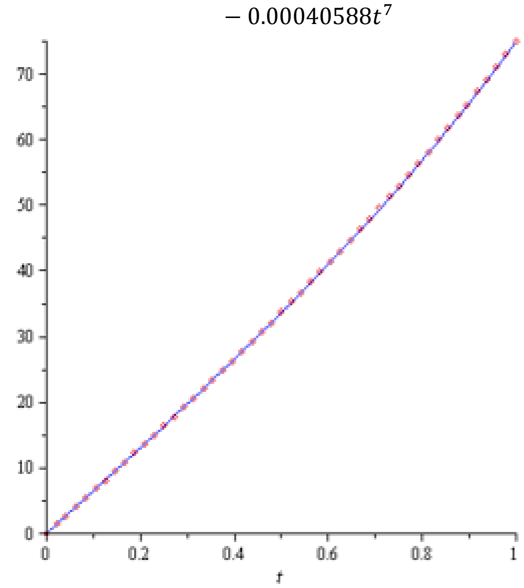
$$u_1(t) = -3.14719289450058300 \times 10^{-306} + 64.37637463t + 4.6750291t^2 - 0.950619t^3 + 6.930721t^4 + 0.107218t^5 - 0.170624t^6 + 0.031898t^7$$

For the second equation, we have;

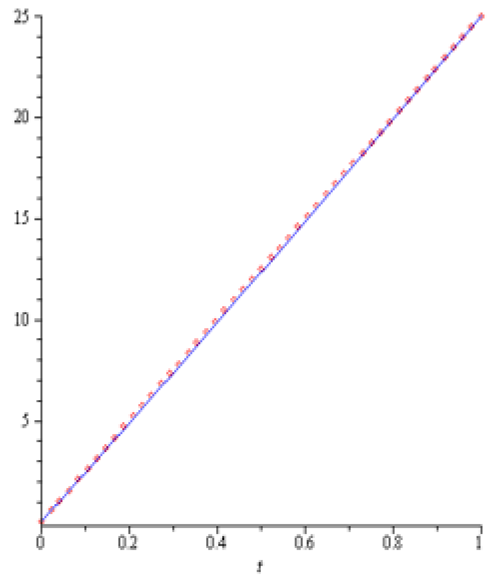
$$u_2(t) = 0.03333333333333332982 + 23.95761023t + 2.020013400t^2 - 1.3327407t^3 + 0.3189426t^4 + 0.045549t^5 - 0.0112941t^6 + 0.0019194t^7$$

For the third equation, we have;

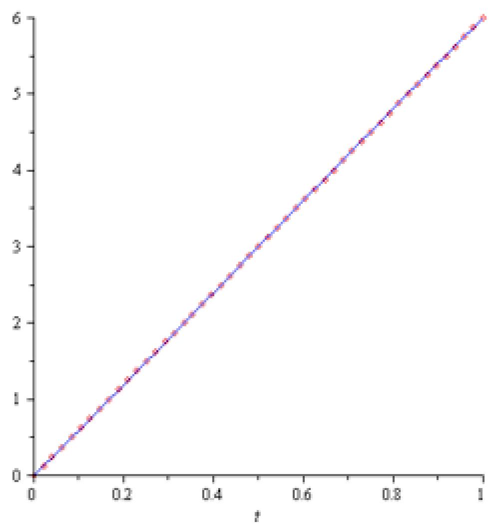
$$u_3(t) = -2.67046585108098486 \times 10^{-307} + 5.896893726t + 0.3022673t^2 - 0.16749935t^3 - 0.0284633t^4 - 0.0006367t^5 - 0.0021561t^6 - 0.00040588t^7$$



2 (a)



2 (b)



2 (c)

Figure 2a-c: Exact and approximate solutions of Example 3 [▫▫▫ Exact --- Bezier]

Example 4: Consider the following systems of disjoint Fredholm-Volterra integro-differential equations with the exact solution; $y_1(t) = t^3 + \sin(t)\,, y_2(t) = 6t - 12t^2, y_3(t) = 2 + 6t + 12t^2, y_4(t) = 32t + 6$ respectively.

$$y'''_1(t) = 5 - \frac{1}{4}t^4 + \int_0^t y_1(s)ds + \int_{-\pi}^{\pi} ty_1(s)ds$$

$$y'''_2(t) = t - 2t^3 + 3t^4 + \int_0^t sy_2(s)ds + \int_0^1 ty_2(s)ds$$

$$y'''_3(t) = -6 - 2t - 3t^2 - 4t^3 + \int_0^t y_3(s)ds + \int_0^1 sy_3(s)ds$$

$$y'''_4(t) = -\frac{1}{2}t^2 + \int_0^t y_4(s)ds + \int_{-\pi}^{\pi} ty_4(s)ds$$

For the first equation, we have the solution;

$$u_1(t) = -4.097408949557910 \times 10^{-308} + 0.9379032803t + 0.081726408t^2 + 0.83333311t^3 - 0.01932527t^4 + 0.00780639t^5 + 0.00024607t^6 - 0.00021904t^7$$

For the second equation, we have;

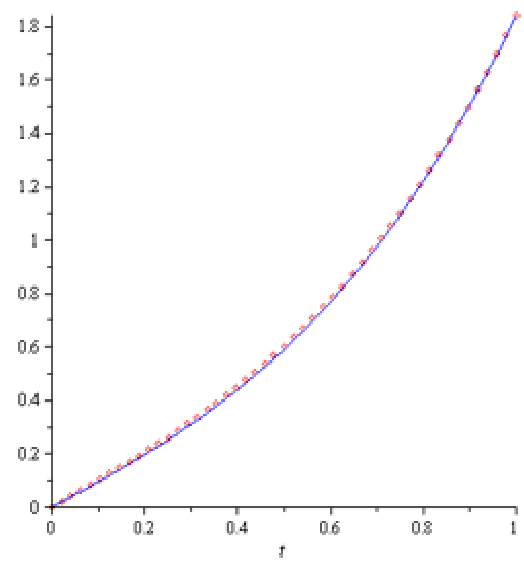
$u_2(t) = 1.33504431510432394 \times 10^{-307} + 6.0000000014t - 12.00000000t^2 + 1.10^{-7}t^3 - 5.89206858 \times 10^{-8}t^4 + 1.10^{-17}t^5 + 1.4 \times 10^{-7}t^6 - 1.10^{-8}t^7$

For the third equation, we have;

$$u_3(t) = 2.0 + 6.00000008t + 11.999999988t^2 - 1.10^{-7}t^6$$

For the fourth equation, we have;

$$u_4(t) = 6.0 + 31.55959744t + 0.8276319t^2 - 0.000000072t^3 - 0.643810t^4 + 0.254335t^5 + 0.002953t^6 - 0.0006990t^7$$



3 (a)

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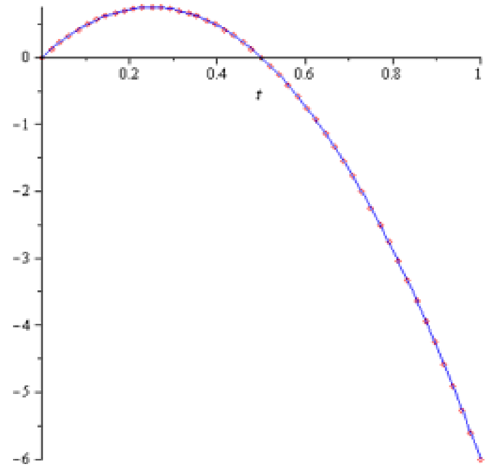
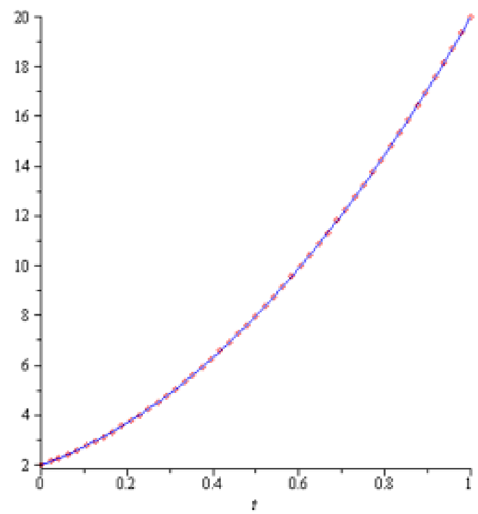
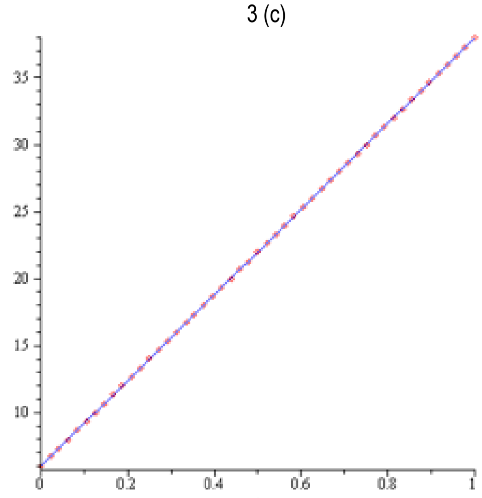


3 (b)



3 (c)



3 (d)

Figure 3a-d: Exact and approximate solutions of Example 4 [▫▫▫ Exact --- Bezier]

Discussion

Table 1 shows the degree of accuracy of this method in terms of absolute errors for $n = 8$ compared to the fixed point iterative (Berenguer, *et al* 2012) and Bernstein operation matrix (Maleknejad, *et al* 2012) methods, while Figures 1-3 show the plots of the exact solution (in red) against the Bezier approximate solution (in blue) for systems of 2-4 FVIDEs in examples 2-4 respectively at $n = 7$. These clearly show that the Bezier control-point-based method is an accurate and effective technique for finding approximate solutions to Fredholm- Volterra integro-differential equations. The computations were done using Maple 16.

Conclusion

In this work, the Bezier control points method has been used successfully to find the approximate solution of systems of disjoint Fredholm-Volterra Fredholm-Volterra integro-differential equations (FVIDEs). The approximate solutions obtained are of high accuracy. Thus, the method is effective

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