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Full Length Research Article

GENERALIZED MHD COUETTE FLOWS IN AN ANNULI: THE RIEMANN-SUM APPROXIMATION APPROACH

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ABSTRACT
In this study, a mathematical analysis is presented for unsteady MHD generalized Couette flow of an incompressible and electrically conducting fluid in an annulus formed by two concentric cylinders of infinite length. The governing equations are obtained and solved using Laplace transform and the Riemann sum approximation method. The results for various ensued parameters on the velocity, skin friction and mass flux have been presented graphically and discussed.

Keywords: Laplace transform, Generalised MHD, magnetic field, annuli, Riemann-sum approximation.

INTRODUCTION
Flow formation of an electrically conducting fluid through a circular pipe in the presence of a transverse magnetic field is encountered in a variety of applications such as Magnetohydrodynamic (MHD) generators, pumps, accelerators, and flow meters (Jha and Apere, 2013). Antimirov and Kolyshkin (1984) studied the unsteady MHD flow in an annular channel with radial magnetic field while Takhar *et.al* (1989) examined the stability of MHD Couette flow in a narrow gap annulus. The annular geometry is widely employed in the field of heat exchangers such as in gas cooled nuclear reactors in which the cylindrical fissionable fuel elements are placed axially in vertical coolants channel within the graphite moderators and the cooling gas is flowing along the channel parallel to the fuel elements (Singh *et. al*, 1997).

Globe (1959) was first to presents an analysis of the fully developed laminar MHD flow in an annular channel. Jain and Mehta (1962) extended the problem by imposing suction/injection on the walls. An exact solution of electrically conducting viscous incompressible flow in an annulus with porous walls under an external radial magnetic field was obtained by Nandi (1973).

The generalised MHD Couette flow was attempted by Agarwal (1965) where he analysed the problem by giving the conditions under which the back flow at the stationary wall may be avoided. Soudalgekar (1966) discussed the temperature field problem in the case of the generalised Couette flow under the action of transverse magnetic field and have found out that the rate of heat transfer at the stationary plate is affected by the magnetic field. In another related investigation, Soudalgekar (1969) considered the effects of both electric and magnetic fields on the flow. Esmaeilpour and Ganji (2008) studied Generalized Couette Flow by He's Methods and Comparison with the Numerical Solution while Makinde and Onyejekwe (2011), presented numerical study of MHD generalized Couette flow and heat transfer with variable viscosity and electrical conductivity. Chauhan *et. al* (2012) presented the effect of thermal-diffusion (Soret) and diffusion-thermo (Dufour) in hydro-magnetic generalized couette flow of a binary mixture of gases in presence of normal applied magnetic field in porous medium. Hazem, Karem and Nabil (2012) studied the effect of porosity on the transient MHD generalized Couette flow with heat transfer in presence of heat source and uniform suction and injection. Animesh (2014), studied generalized magnetohydrodynamic Couette Flow of a binary mixture of viscous fluids through a horizontal channel under Soret effect. Singh, Joshi, and Srinivasa (2017), studied unsteady MHD Generalized Couette Flow in a rotating channel with induced magnetic field, hall current and periodically magnetized walls. Taiwo and Michael (2018) investigated the combined effect of radial magnetic field and viscous dissipation on entropy generation in horizontal co-axial cylinders of generalized Couette flow while Mollah *et. al* (2019) investigated numerically the MHD generalized Couette flow and heat transfer on Bingham fluid through porous parallel plates with ion-slip and Hall current.

Despite the amount of studies that has been carried out on the annular geometry under different conditions, little or no work seems to have been carried out on the generalised MHD Couette flow in an annuli in which the pressure-induced flow is supposed to be superimposed on simple shear flow in classical hydrodynamics. It is the object of this work therefore, to present a solution for the generalised MHD Couette flow in an annulus formed between two concentric horizontal cylinders of infinite length when the outer cylinder has been set into impulsive motion.

MATERIALS AND METHODS

Mathematical Formulation
We consider the motion of a viscous, incompressible and electrically conducting fluid between two horizontal concentric cylinders of infinite length (see Fig. 1). The outer cylinder is impulsively started with a uniform velocity along the flow direction, while the inner cylinder is kept stationary. A constant pressure gradient exists in the flow direction. The fluid flow between the two the cylinders in the presence of magnetic field acting perpendicular to the flow direction. The z-axis is assumed to be on the axis of the cylinder in the horizontal direction and r'-axis is on the radial direction. We assume that the magnetic Reynolds number is very small which correspond to negligible induced magnetic field compared to the externally applied field. Furthermore, the uniform magnetic field B is constant $B \equiv (0,0,B_0)$ and is considered as the total magnetic field acting on the fluid. At $t' \leq 0$, the fluid and the cylinders are assumed to be at rest. When $t' > 0$ the outer cylinder is set into motion with a uniform velocity U, and the inner cylinder remains at rest. Since the cylinders are of infinite length and the flow is fully developed, all physical variables are functions

of r' and t' only. It is also assumed that no applied and polarization voltage exists. Under these assumptions, the mathematical model governing the present physical situation is given as;

$$\frac{\partial u'}{\partial t'} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon[\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}] - \frac{\sigma B_0^2}{\rho}u', \quad (1)$$

Where ρ is the density, υ is the kinematic viscosity and σ is the electrical conductivity. Equation (1) is valid when the magnetic field is fixed relative to the fluid. If the magnetic field is fixed relative the moving cylinder, can be written as;

$$\frac{\partial u'}{\partial t'} = -\frac{1}{p}\frac{\partial p}{\partial z} + \upsilon[\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}] - \frac{\sigma B_0^2}{\rho}\left[u' - KUt'^m\right], \quad (2)$$

Equations (1) and (2) can be unified to obtain;

$$\frac{\partial u'}{\partial t'} = -\frac{1}{p}\frac{\partial p}{\partial z} + \upsilon[\frac{\partial^2 u'}{\partial r'^2} + \frac{1}{r'}\frac{\partial u'}{\partial r'}] - \frac{\sigma B_0^2}{\rho}[u' - KU] \quad (3)$$

Where

$$K = \begin{cases} 0 & \text{when } B_0 \text{ is fixed relative to the fluid} \\ 1 & \text{when } B_0 \text{ is fixed relative to the moving cylinder} \end{cases}$$

The initial and boundary conditions for the problem are;

$$t' \leq 0:\ u' = 0 \text{ for } 0 \leq r' \leq b,$$

$$t' > 0: \begin{cases} u' = 0 & at \quad r' = a \\ u' = Ut'^m & at \quad r' = b \end{cases} \quad (4)$$

The flow described by equations (3) and (4) is the general representation of the velocity for the MHD Couette flow between two infinite concentric cylinders of any conducting medium due to the motion of the outer cylinder. In order to analyze the flow behavior, we have considered the case when $n = 0$, which correspond to the impulsive motion. Introducing the following non-dimensional quantities

$$r = \frac{r'}{a}, \lambda = \frac{b}{a}, t = \frac{t'\upsilon}{a^2}, M^2 = \frac{\sigma B_0^2 a^2}{\rho\nu}, u = \frac{u'}{U}, P = \frac{1}{\rho}(-\frac{\partial p}{\partial z})\frac{a^2}{U\nu}. \quad (5)$$

Where a and b are the radii of the inner and outer cylinders respectively. λ, is the annulus dimension ratio and is fixed as 3 in this paper, M is the Hartman number, which is a measure of the strength of the applied magnetic field. Eqn. (3) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - M^2[(u - K)] + P \quad (6)$$

Subject to the following dimensionless initial and boundary conditions

$$t \leq 0:\ u = 0, \text{ for } 1 \leq r \leq \lambda,$$

$$t > 0: \begin{cases} u = 0 & at \quad r = 1 \\ u = U & at \quad r = \lambda \end{cases} \quad (7)$$

The solution of equation (6) is obtained by Laplace transform Technique with corresponding boundary conditions as follows;

$$\bar{u} = C_1 I_0(\delta r) + C_2 K_0(\delta r) - \frac{A}{s\delta^2} \quad (8)$$

Equation (8) is in Laplace domain, hence the need to get its inverse in order to determine the velocity of the fluid flow in time domain. To achieve this, numerical procedure used in (Jha and Apere, 2010) is employed. In this method, functions in the Laplace domain s can be inverted to the time domain T as follows:

$$u(r,t) = \frac{\exp(\epsilon T)}{T}\left[\frac{1}{2}\bar{u}(r,\epsilon) + Re\sum_{k=1}^{N}\bar{u}\left(r, \epsilon + \frac{ik\pi}{T}\right)(-1)^k\right] \quad (9)$$

where Re refers to the 'real part of', $i = \sqrt{-1}$ is imaginary number, N is the number of terms used in the Riemann-sum approximation and ϵ is the real part of the Bromwich contour that is used in inverting Laplace transforms. The Riemann-sum approximation for the Laplace inversion involves a single summation for the numerical process. Its accuracy depends on the value of ϵ and the truncation error dictated by N. According to Tzou [13], the value of ϵ must be selected so that the Bromwich contour encloses all the branch point. For faster convergence the quantity $\epsilon T = 4.7$ gives the most satisfactory results since other tested values of ϵT seem to need longer computational time.

Skin-Friction and Mass Flux

The skin friction τ is obtained by differentiating equation (9) with respect to r

$$\tau = \frac{du}{dr} \quad (10)$$

The mass flux of the fluid flow through the concentric cylinders of an infinite extent is obtained as;

$$Q(r,s) = \int_1^{\lambda} r.\bar{u}(r,s)ds = \frac{1}{\delta}\left[C_1\left(I_1(\delta) - \lambda I_1(\lambda\delta)\right) + C_2\left(\lambda K_1(\lambda\delta) - K_1(\delta)\right)\right] - \frac{1}{2}\frac{A}{s\delta^2}(A^2 - 1) \quad (11)$$

Steady State Solution

In order to validate the accuracy of the Riemann-sum approximation method, we set out to find the solution of the steady state, which should coincide with the transient solution at large time. The equation for the steady state velocity is obtained by setting $\frac{\partial u}{\partial t}$ in Equation (6) to zero;

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - M^2 u = -P - M^2 K \quad (12)$$

And its solution is given as;

$$u_s = C_3 I_0(Mr) + C_4 K_0(Mr) + A_2 \quad (13)$$

The steady state skin friction at the outer surface of the inner cylinder τ_1 and that of the inner surface of the outer cylinder τ_λ is obtained by differentiating equation (13) as;

$$\frac{du_s}{dr} = [C_3 I_1(Mr) - C_4 K_1(Mr)]M \quad (14)$$

Where $A_2 = \frac{P}{M^2} - K, A_3 = K_0(M)I_0(M\lambda) - I_0(M)K_0(M\lambda),$

$$C_3 = \frac{[K_0(M)(1 - A_2) + A_2 K_0(M\lambda)]}{A_3}, \quad C_4 = \frac{[I_0(M)(A_2 - 1) + A_2 I_0(M\lambda)]}{A_3}$$

RESULTS AND DISCUSSION

In order to study the physical aspect of the problem, semi analytical solution is obtained by the combination of Laplace transform technique and the Riemann Sum Approximation. Expressions for velocity, skin-friction, mass flux and those for steady state are derived and presented graphically in Figs. 2 - 15 for K= 0, K =1; $\lambda = 3, P = -5$ and 5. For the pressure increasing in the direction of flow $P > 0$, the velocity increases as the time increase, while reverse trend is observed in the case of pressure decreasing in the direction of flow $P < 0$. This is evident from Figs.2 and 3. Fig.4 and 5 show the effect of M for $t = 0.2$. It is observed that, increase in M decreases the velocity in the case of positive P while reverse in the case of negative P. The effect is more prominent when $K = 0$ in the case of positive P and when $K = 1$ in the case of negative P. Fig. 6 and 7 depict the effects of M and t on the skin-friction at the outer surface of the inner cylinder τ_1. It is

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observed that, increase in t increases the skin-friction and it decreases with increase in M in the case when $K = 0$ while it increases in both increase in M and t. Reverse behavior is seen in the case of positive P particularly when $K = 0$.

Fig. 8 and 9, show the effects of P, M and t on the skin-friction at the outer surface of the inner cylinder τ_1. It is observed that, the skin-friction increases as t increase for $P > 0$, while reverse is the case for $P < 0$. Worthy of note is the absence any effect when $P = 0$ and $K = 0$. Also, increase in Mincreases the skin-friction for $P < 0$ and $K = 0$ while reverse is the case when $P > 0$.
Fig. 10 and 11 depict the effects of M, P and t on the skin-friction at the inner surface of the outer cylinder τ_λ. It is observed that, τ_λ increases with decrease in t and increases with increase in M in the case of $K = 0$ while it increases with decrease in both M and t. This behaviour is seen when P is positive (Fig.10). While in the case when P is negative (Fig. 11), τ_λ increases with increase in both M and t for $K = 0$, while decreases with increase in M for $K = 1$.
Figs. 12 show the effects of M and t on Q. It is seen that Q increases with increase in both M and t. The effects is more glaring when the magnetic field is fixed with the moving fluid in the middle of the annulus. Figs. 13 show the effects of P and M on Q. It is seen that Q increases with increase in both P and M. The effects is more glaring when the magnetic field is fixed with the moving fluid in the middle of the annulus.

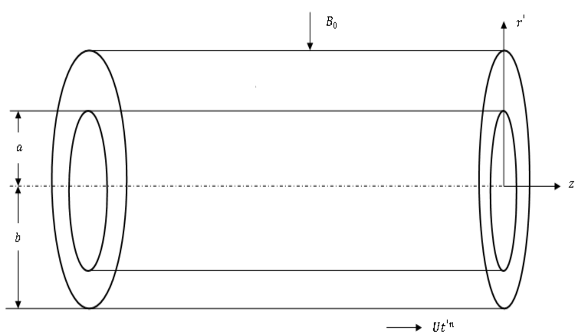

Fig.1: Schematic diagram of the problem

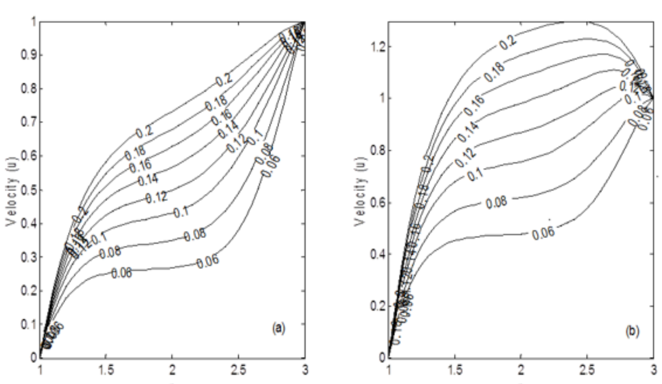

Fig. 2: Velocity profile showing the effect of time t for $K = 0$ and $K = 1$, represented by a and b respectively and $P = 5$.

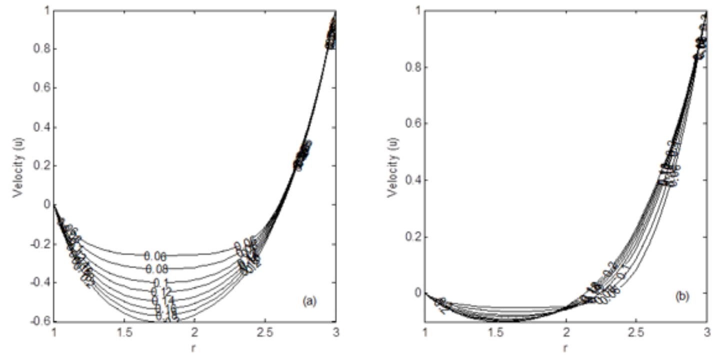

Fig.3: Velocity profile showing the effect of time t for $K = 0$ \$ and $K = 1$ by a and, represented respectively and $P = -5$.

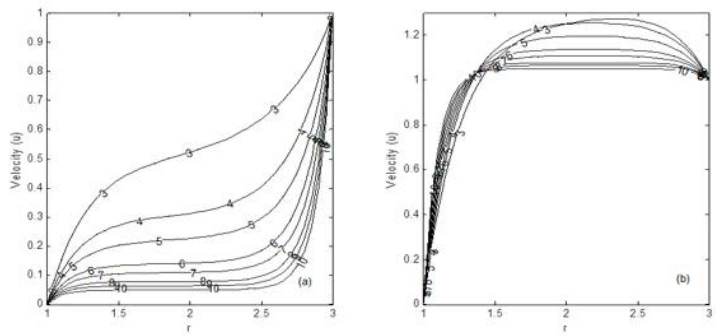

Fig. 4: Velocity profile showing the effect of Hartmann number M for $K = 0$ and $K = 1$, represented by a and b respectively and $P = 5$.

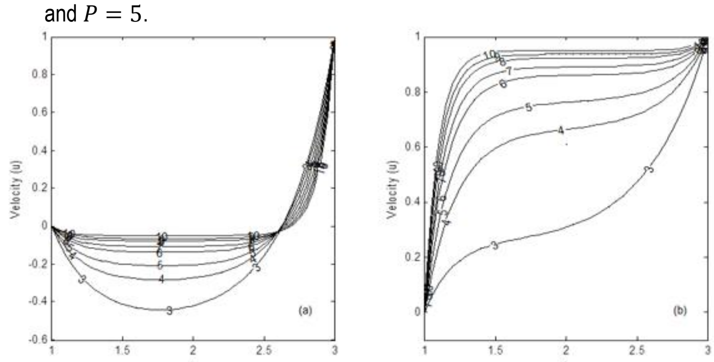

Fig. 5: Velocity profile showing the effect of Hartmann number M for $K = 0$ and $K = 1$, represented by a and b respectively and $P = -5$.

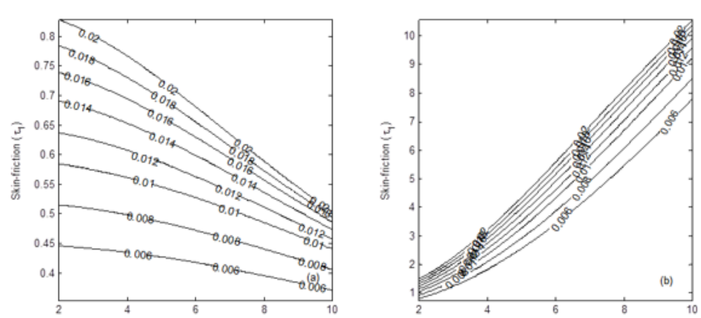

Fig.6: Variation of skin-friction with M and t at the outer surface of the inner cylinder for $P = 5$, $K = 0$ and $K = 1$.

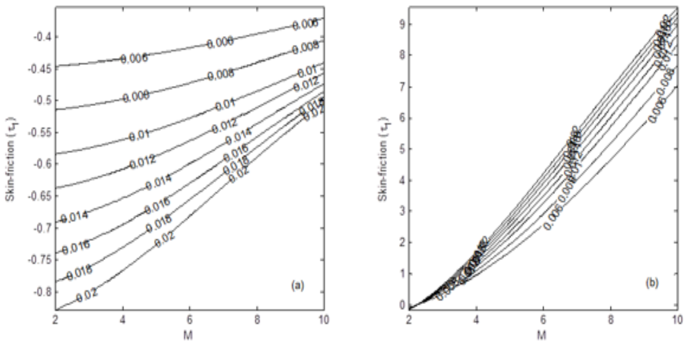

Fig.7: Variation of skin-friction with M and t at the outer surface of the inner cylinder for $P = -5$, $K = 0$ and $K = 1$.

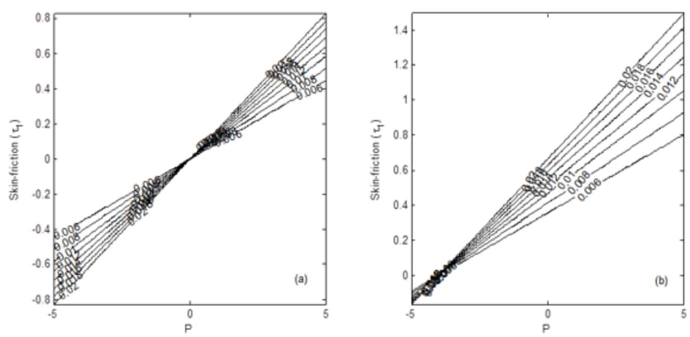

Fig.8: Variation of skin-friction with P and t at the outer surface of the inner cylinder for $P = 5$, $K = 0$ and $K = 1$

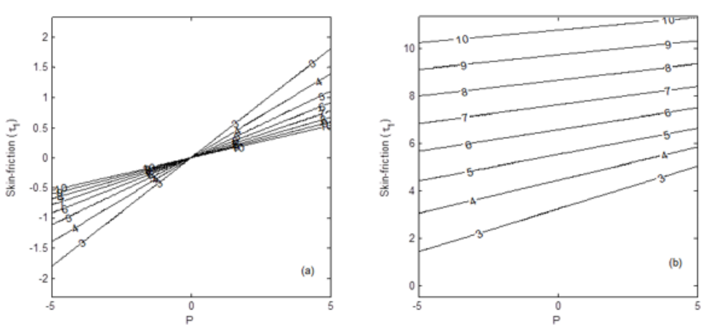

Fig.9: Variation of skin-friction with P and t at the outer surface of the inner cylinder for $P = -5$, $K = 0$ and $K = 1$.

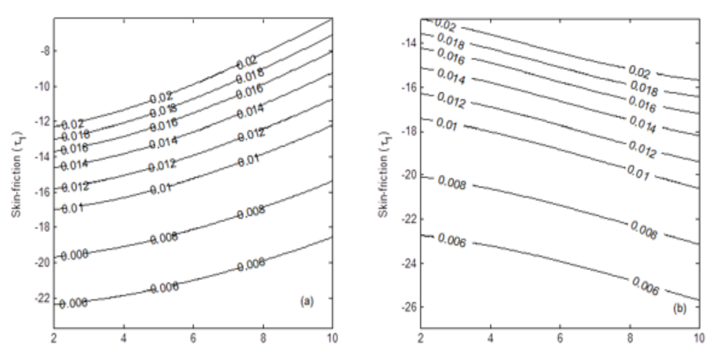

Fig.10: Variation of skin-friction with M and t at the inner surface of the outer cylinder for $P = 5$, $K = 0$ and $K = 1$

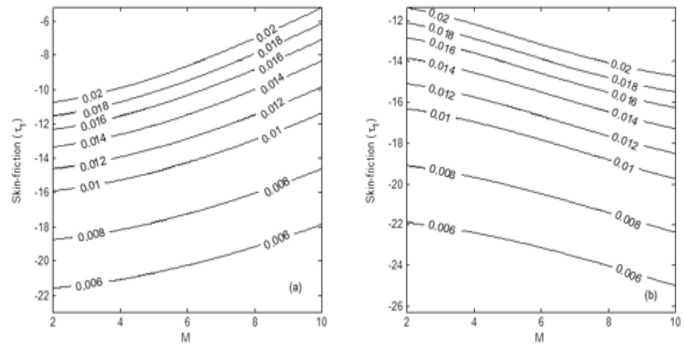

Fig.11: Variation of skin-friction with M and t at the inner surface of the outer cylinder for $P = -5$, $K = 0$ and $K = 1$

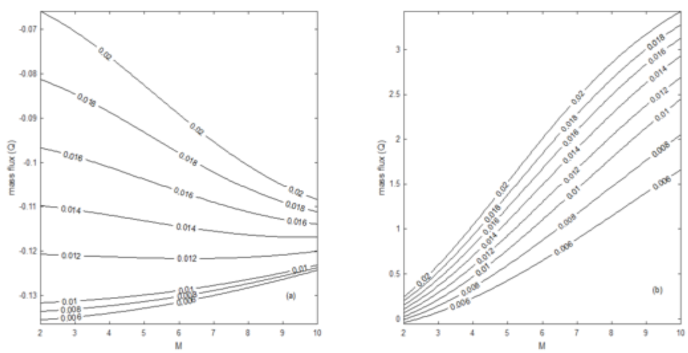

Fig.12: Variation of mass flux (Q) with M and t at for $P = 5$, $K = 0$ and $K = 1$.

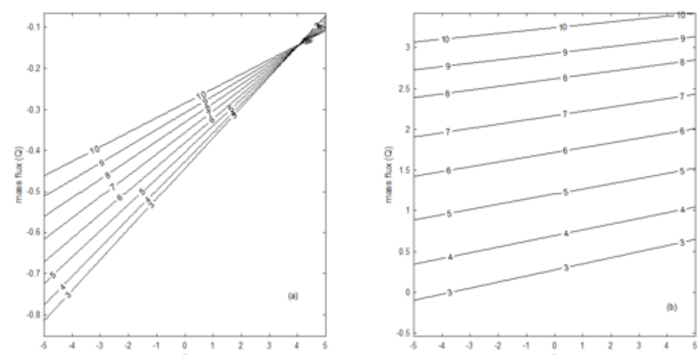

Fig.13: Variation of mass flux (Q) with M and t at for $P = -5$, $K = 0$ and $K = 1$

Conclusion

Generalized MHD Couette flow of a viscous incompressible electrically conducting fluid in an annuli when the outer cylinder is impulsively set into motion is considered. The Riemann-sum approximation method is used to invert the Bessel equation obtained through the Laplace transform technique into the time domain. The effects of Hartmann number M, the imposed pressure P and time t is studied. It is observed that the velocity decreases with increase in Hartmann number M when the magnetic field is fixed relative to the fluid; this is expected, as the magnetic field has a retarding influence on the flow fields. Conversely, an increase in velocity is noticed with increase in M when the magnetic field is fixed relative to the moving outer cylinder. The skin friction is discovered to increase with t at all times, while decreases with M on the outer surface of the inner cylinder when $K = 0$ and on the inner surface of the outer cylinder when $K = 1$. However, the skin friction increases with M on the outer surface of the inner cylinder when $K = 1$ and also on the inner surface of the outer cylinder when $K = 0$.

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