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Full Length Research Article

# STRUCTURE TO POLYNOMIAL FUNCTORS IN ORTHOGONAL CALCULUS I

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**ABSTRACT**
In the study, algebraic structures and maps (category theory, morphisms and functors) that are inherently tied to the calculus of functors (orthogonal calculus) were explored. We emphasized on linear polynomial functors and generalized it to the n- polynomial functors as in the algebraic and topological settings where the topological settings talked about the Goodwillie and the Weiss case.

**Keywords:** Calculus of Functors; Section; Retract; Fibration; Category; Pushout squares, Pullback Squares.

## 1. INTRODUCTION

In mathematics especially in the algebraic and differential topology, the functor calculus i.e. the Goodwillie calculus of homotopy functor, the Weiss calculus of functor and finally the embedding calculus is a technique to studying functors. These functors are well studied by approximating them with sequence of simpler functors. These sequence of approximation is almost same as the Taylor series of smooth function.

There are many objects in algebraic and differential topology which can be seen as functors. They maybe functors although but it's always difficult to analyze directly, so we think of replacing them with simpler functors which are sufficiently good approximation for the functor in question.

The calculus of functors was developed by a mathematician known as Thomas Goodwillie. Goodwillie came out with a series of three papers on the calculus of functors (Goodwillie, 1990) (Goodwillie, 1991) (Goodwillie,2003) in the 1990s and 2000s. He had his inspiration from the work done by Eilenberg and Mac Lane on functors in the 1940s (Eilenberg and MacLane, 1945).

This calculus of functors is known as the Goodwillie calculus of homotopy functor which has been the source of motivation for the other calculus of functors.

Micheal Weiss calculus of functor emerged after the papers of Goodwillie were published which is known as the orthogonal calculus of functors, due to (Weiss, 1995), and this theory is closely related to or he had his inspiration from the Goodwillie calculus of homotopy functor. The orthogonal calculus of homotopy functor is a beautiful tool for calculating the homotopical properties of functors from the category of vector spaces to pointed spaces or any space enriched over Top∗. With the Weiss calculus we consider covariant functor from the category of vector spaces (finite dimensional) with an inner product to the category of spaces $(Top_*)$ instead of functors from spaces to spaces as defined by Goodwillie (Weiss, 1995). There are intriguing examples of such functors and they include classical objects in algebraic and geometric topology such as :

i. $BAut(V)$
ii. $BTop(V)$
iii. $\Omega^V(S^V \wedge X)$

Category of such functors from vector spaces to spaces and natural transformations between them will be called $\xi_0$. Orthogonal calculus is based on the notion of *n*-polynomial functors (vector spaces at very high dimension), which are well-behaved functors in $\xi_0$ and which preserves weak equivalences as well. With these *n*-polynomial functors one can often infer the value at some vector spaces from the values at vector spaces of higher dimension (Barnes and Oman, 2013). In general sense, orthogonal calculus approximates a functor (locally around) via polynomial functors (approximate into sequence of simpler functors that are homotopy equivalent to the functor in question) and attempts to reconstruct the global functor from the associated ‘infinitesimal’ information. The orthogonal calculus splits a functor, F, into a Taylor tower of fibrations, where we can think of the $n-th$ fibrations to consist of an arrow (map) from the n-polynomial approximation of F to the $(n-1)-$ polynomial approximation of F. The homotopy fiber or layer (the difference between n-polynomial and $(n-1)-$ Polynomial approximation) of this map is then an n homogeneous functor and is classified by an $O(n)-$ spectrum up to homotopy which is usually denoted as $D_nF$.

## 2. Priliminaries

### Algebraic structures and maps.

Before looking at polynomials in orthogonal calculus of functors, we will highlight on the category theory and functors that are inherently tied to calculus of functors and which will commonly be encountered.

### 2.1 Category Theory

**Defintion 2.1.** A category $\xi$ consist of

i. A collection *A, B* $\in$ $\xi$ where A and B are objects in $\xi$
ii. For all A, B ∈ obj (ξ), a collection ξ (A, B) of maps (arrows) from the object A to the object B.
iii. To every map there exist two objects, its source and target. If f is a map with a source A and target B, then we indicate this by $f : A \to B$, (Knighten, 2007).

For every object A there exist an associated identity map which is written as $1_A : A \to B$. Further if $f : A \to B$ is a map from an object *A* to the object *B* and $g : B \to C$, is also a map from object *B* to object *C* then there will exist a composition $gf : A \to C$, which will satisfy the following relations.

a) If $f : A \to B$, $g : B \to C$, and $h : C \to D$ then $h(gf) = (hg)f : A \to D$ (Associativity condition).
b) If $f : A \to B$, then $f1_A = f = 1_B f$ (Identity Morphism).

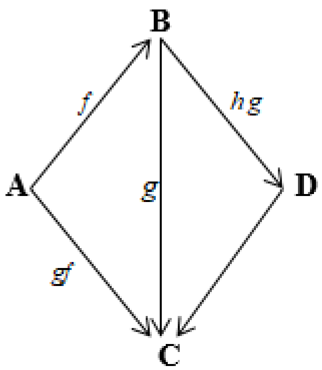


**Figure 1a**: Associativity relation of category theory

Relations such as the associativity and the identity morphism are denoted by saying the figures above commutes.

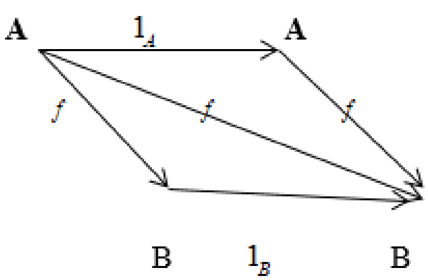


**Figure 1b**: Identity morphism of category theory.

### 2.2. Sub- Categories

Definition 2.2. A category $\varepsilon$ is a subcategory of category $\xi$ provided :

i. Each obj (ε) is in obj (ξ)
ii. $f \in \varepsilon$ (A, B), implies $f \in \xi$ (A, B) i.e. if *f* is a map in the sub-category ε implies *f* is also a map in the main category ξ.
iii. $f : A \to B$ and $g : B \to C$ in ε, implies *gf* is a composition of f following g in ξ. i.e. composition of maps in subcategories also holds in the main category.
iv. If 1A is the morphism (identity) for A in ε, then $1_A$ is also a morphism (identity) for A in ξ. (Knighten, 2007)

### 2.3. Special Morphism

Relations among morphisms are often shown with diagrams that commutes, with points or block letters to represent objects and arrows to represent maps or arrows.
The identity morphisms is a natural map that every object have. Aside the natural map (identity) that exist in every type theory, there exist other different maps that are useful and interesting to study as well. i.e. Morphisms can have any of the following properties.

**Isomorphism**
If $f : A \to B$ is a map from the source A and the target B with $f^{-1}f = 1_A$ and $ff^{-1} = 1_B$ as inverses, (Knighten, 2007).
If there exist an inverse for f, then it should be unique, to justify the uniqueness of the notation $f^{-1}$. To see that f is unique up to isomorphism lets consider *g* and *h* to be inverses of f, then
$g = 1_A g = hfg = h1_B = h$.
We will denote an isomorphism with the symbol '$\cong$ ' and write $f : A \xrightarrow{\cong} B$ to show an isomorphism of the morphism *f*, and will denote an isomorphism of A and B as $A \cong B$ and say A is isomorphic to B" if A and B are isomorphic to each other.

**Homomorphism**
If M and N are two abelian groups.
A mapping $\varphi : M \to N$ is called homomorphism if for all $x, y \in M$ $\phi(xy) = \phi(x)\phi(y)$ (Nkrumah *et al.,* 2019).
To a map of spaces we can associate a homomorphism of groups such that compositions of maps yield compositions of homomorphisms of groups, (Zigli *et al.,* 2017).

**Endomorphism**
A morphism $g : B \to C$ is an endomorphism if $B = C$. *End* (*B*) denotes the group of endomorphisms of B.

**Automorphism**
An endomorphism that guarantee a return inverse or also an isomorphism is known as an automorphism. i.e. is an endomorphism that has left and right inverses. The automorphism class of the object C is the group of all automorphism of C. And is usually represented with *Aut* (*C*).

**Section And Retract**
The definition of an isomorphism can be separated into two parts i.e. isomorphism have both left and right inverses, which is the same as we saying an isomorphism has both a section and a retraction.

**Section**
For any map $f : A \to B$, a section of f is map $s : B \to A$ such that $f \circ s = 1_B$. A section is called a right inverse (Panangaden, 2012).

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**Retract**

For any map $f : A \to B$, a retraction of f is a map $r : B \to A$, such that $r \circ f = 1A$. A retract is also called a left inverse (Panangaden, 2012). In any category a section is a monic and a retraction is an epic, but the converses are not always true.

**Note**: Morphism that is a section and at the same time a retraction is also called an isomorphism.

**Epimorphism (Epic)**

In any category, the map $e : A \to B$ is an epimorphism or epic morphism in a way that, $\forall, f, g : B \to C$, $f \circ e = g \circ e$ implies $f = g$. The equation $f \circ e = g \circ e$ implies f and g are two morphism with Source B and target C, (Panangaden, 2012).

$$\begin{tikzcd} A \arrow[r, "e"'] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

**Figure 3**: Diagrammatic explanation of anepimorphism

**Monomorphism (Monic)**

In any category, the map $m : B \to C$ is said to be monomorphism or monic morphism for the fact that $\forall f, g : A \to B$, $m \circ f = m \circ g$ implies $f = g$ (Panangaden, 2012).

The equation $m \circ f = m \circ g$ implies f and g are two maps with the source A and the same target point B.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "m"] & C \end{tikzcd}$$

**Figure 4**: Diagramatic explanation of monomorphism

## 2.4. Dual Category

Definition 2.3. C been a category, would imply $C^{op}$ is its dual category and it is defined as follows (Knighten, 2007)

i. They objects in $C^{op}$ is exactly the same objects as to that of C.

ii. The maps of $C^{op}$ are the reversed version of arrows of C, i.e. for every arrow $f : A \to B$, there exist a morphism $f^{\#} : B \to A$ in $C^{op}$.

iii. The composition of arrows $g^{\#} \circ f^{\#}$ in $C^{op}$ is nothing but $(f \circ g)^{\#}$.

**Product in a category**

A product diagram for the objects A and B consists of an object p and morphisms $A \xleftarrow{\pi_1} P \xrightarrow{\pi_2} B$ satisfying the following universal mapping property :

Given any diagram $A \xleftarrow{x_1} P \xrightarrow{x_2} B$, there exist a unique $u : X \to P$ such that the following diagram commutes :

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

We denote the product of A and B as $A \times B$ (Robin, 2020).

**Coproduct in a category**

Is a concept describing the (categorical anlogues of the) construction of a direct sum of modules or a decrete union of sets in the language of morphism. Let $A_i, i \in I$ be an indexed family of objects in a category $\Re$. An object S, together with morphisms $\sigma_i : A_i \to S$ is called a coproduct of the family $A_i, i \in I$ if for any family of morphism $\alpha : A_i \to X, i \in I$, there exist a unique morphism $\alpha : S \to X$ such that $\sigma_i \alpha = \alpha, i \in I$.

The morphisms $\sigma_i$ are called imbeddings of the coproduct, the coproduct is denoted by $\prod_{i \in I}^{*} A_i(\sigma_i)$ (Hazewinkel, 2013).

**Pullback**

In any category C, a pullback of arrows f, g with $cod(f) = cod(g)$

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

Consist of arrows

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \\ A & \end{tikzcd}$$

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Such that $f\rho_1 = g\rho_2$, and universal with this property. Ie given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $fz_1 = gz_2$, there exist a unique $u : Z \to P$ with $z_1 = \rho_1 u$ and $z_2 = \rho_2 u$ (Awodey, 2010).

$$
\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[ddr, "z_1"'] \arrow[drr, "z_2"] & & \\
& A \times_c B \arrow[r, "\rho_2"] \arrow[d, "\rho_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}
$$

**Pushout**

For $B =$

$$
\begin{tikzcd}
& 1 \arrow[dl, "u"'] \arrow[dr, "v"] & \\
2 & & 3
\end{tikzcd}
$$

We have that the object of $\mathbb{C}^B$ are diagrams in $\mathbb{C}$ of the form

$$
\begin{tikzcd}
& X_1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_2 & & X_3
\end{tikzcd}
$$

And arrows to the diagonal functor are pairs of arrows h and k making the diagram

$$
\begin{tikzcd}
& X_1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_2 \arrow[dr, "h"'] & & X_3 \arrow[dl, "k"] \\
& Z &
\end{tikzcd}
$$

commute. Equationally : $h \circ f = k \circ g$. The universal such an arrow (if it exists) is called the pushout of f and g. Spelling out the details, it consist of a pair of arrows p and q in $\mathbb{C}$ such that $p \circ f = q \circ g$

$$
\begin{tikzcd}
& X_1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_2 \arrow[dr, "p"'] & & X_3 \arrow[dl, "q"] \\
& P &
\end{tikzcd}
$$

and with the universal property that for every commuting square there exists a unique arrow $r$ such that $r \circ p = h$ and $r \circ q = k$ (turi, 2001). Diagrammatically:

$$
\begin{tikzcd}
& X_1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_2 \arrow[dr, "p"'] \arrow[ddr] & & X_3 \arrow[dl, "q"] \arrow[ddl] \\
& P \arrow[d, dashed] & \\
& Z &
\end{tikzcd}
$$

The diagram is then called a pushout square.

## 3. Functors

Functors or covariant functor is morphism or an arrow that preserves the structures between categories.

Functors are now applied almost everywhere in modern mathematics to relate various categories.

**Definition 3.1**. A covariant functor $F : C \to D$ is a map that preserves the structures that exist between categories C and D and also associates each object A in category C to an object F(A) in category D; and each morphism $f : A \to B$ in category C to a morphism $F(f) : F(A) \to F(B)$ in D, such that $F(1_A) = 1_{F(A)}$ for every object A in category C; and $F(g \circ_c f) = F(g) \circ_D F(f)$ for every map $f : A \to B$ and $g : B \to C$ for which compositions $\circ_C$ and $\circ_D$ are defined in categories C and category D (Philips, 2010).

**Functor Diagram**

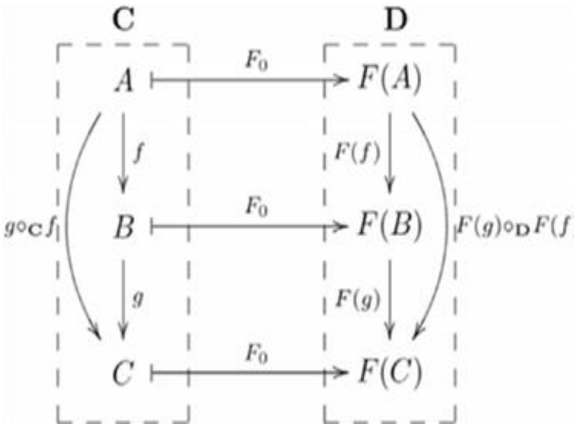


**Figure 5**: Diagramatic explanation of a functor

Where we represent the categories with dashed rectangles, and the functors are represented with $F_0$ and the functor arrows between morphisms are omitted. Composition of Functor and functor isomorphism are defined analogously to morphisms (above). I.e. the functor composition of $F : C \to D$ and $G : D \to E$ is the functor $G \circ F : C \to E$ sending all the objects A in category C to objects $G \circ F(A) \in E$ ; and morphisms $f : A \to B \in C$ to morphisms, $G \circ F(f) : G \circ F(A) \to G \circ F(B)$ such that identity morphism and composition holds. I.e. $G \circ F(1_A) = 1_{G \circ F(A)}$ and $G \circ F(g \circ_C f) = (G \circ F(g)) \circ_E (G \circ F(f))$.

### 3.1. Contravariant Functor

A functor (contravariant) from category C to category D is a functor from Cop to D.Also we can say, F is a contravariant functor if

i. F sends objects $A \in ob(C)$ to object $FA \in ob(D)$
ii. F sends morphisms $f \in C(A, B)$ to morphisms $Ff \in D(FB, FA)$
iii. The identities are preserved
iv. $F(f \circ g) = Fg \circ Ff$

### 3.2. Forgetful/ Underlying Functors

A functor is defined as an underlying functor or forgetful functor if it drop some or all the input structure or properties. Examples of forgetful functors are (Leinster, 2016)

i. The functor $U : Top* \to Top$ which embeds the category of pointed topological space into the category of topological space by forgetting that the topological space is pointed.
ii. The functor $U : Group \to Set$ which forgets that a group have more structure than just the underlying set it captured or remembered.
iii. Similarly there exist a functor $U : Ab \to Grp$ defined by U(A) = A for A been an abelian group. This functor forgets the property that abelian groups are abelian. The forgetful functors in this example forget the property on the objects.

## 4. RESULTS AND DISCUSSION

### 4.1. Polynomial Functors

Polynomial functor is just categorification of `polynomial functions'. Polynomial functors has appeared to be very important in physics, also in mathematics with special areas like topology (Bisson and Joyal, 1995), (Pirashvili, 2000), and in algebra (Macdonald, 1998) and also and it route in mathematical logic (Girard, 1988) ,(Moerdijk and Palmgren, 2000) and computer science (theoretical), (Jay and Cockett, 1994), (Abbott *et al.,* 2003), (Setzer and Hancock, 2005) and useful in the representation theory of symmetric groups (Macdonald, 1998). Depending on the properties of polynomial functions one takes as guideline for the Categorification, different notions result which might deserve to be called polynomial functors.

A continuous function $f : R \to R$ is linear if $f(a + b) = f(a) + f(b)$ for all $a, b \in R$. To be precise, we might think of a function been affine linear if

$f(a + b) - f(a) - f(b) + f(0) = 0$ equivalently $f(a + b) - f(0) = (f(a) - f(0)) + (f(b) - f(0))$.

One of the nice properties of functions of real numbers is the property that $f(a + b) - f(a) - f(b) + f(0) = 0$ implies that $f$ is actually an affine linear polynomial in the sense that if we take $f(x) = mx + c$ for some real numbers m and c. Conversely, a function $f(x) = mx + c$ is known as polynomial of degree 1.

### 4.2. Polynomial linear functor

**Algebraic settings**

The study of group theory naturally leads to problem of finding elements of a group that belongs to cyclic subgroups of the group. (Ezearn and Obeng-Denteh, 2015)

**Definition 4.1**. Lets think of $C$ and $D$ to be an Abelian group, $F$ is additive iff

a) Firstly it takes the no object in C to the no object in D. i.e. $F(0_C) = 0_D$.
b) Secondly if it preserves finite product or co-product i.e. $F(A+B) \xrightarrow{\cong} F(A) + F(B)$

**Example 4.1.** Given C and D as categories which are both abelian. Thus we can think of C and D as abelian categories of modules over some commutative rings. $(Mod_R)$. A covariant functor $F : C \to D$ is additive if it respects the enrichment of C and D in abelian groups. If we look at $Hom(A, B) \xrightarrow{F} Hom(FA, FB)$ is an Abelian group homomorphism for every A and B. i.e. The covariant functor F is an enriched over the category of Abelian group.

**Topological – Algebraic Settings**

**Definition 4.2.** Lets think of C to be pointed category with co-products, and D be an Abelian group. $F : C \to D$ Is additive if it preserves

i. $F(0_C) = 0_D$ and

ii. $F\left(A\coprod B\right) \xrightarrow{\cong} F(A) + F(B)$

Example 4.2. The reduced homology $H_* : Top_* \to grAb$. Ie $H(*) = 0$ and $H(A \vee B) \cong H_*(A) + H_*(B)$ satisfies this property.

**Remark.** The additivity of the reduced homology group is captured by

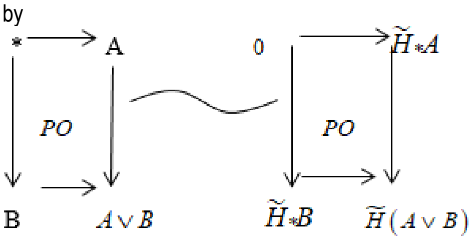


**Figure 6** : Pushout squares of pointed category with coproduct and abelian groups

Which preserves this kind of pushout ? Hence is additive when the reduced homology group preserves this kind of pushout. Homology also has interesting property when applied to dfferen types of pushout squares produces the Mayer-victoris sequence

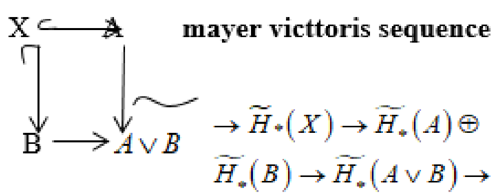


**Figure 7**: homotopy pushout to mayer victoris sequence

This is a stronger property than pure additivity condition. Hence $H_*$ is excisive since it has the Mayer-victoris sequence for homotopy pushout squares.

**Topological Settings (Goodwillie Case)**

**Example 4.3**. Consider the homotopy functor $F : Top* \to sp$ (f weq $\Rightarrow$ Ff weq). F is additive (reduced degree ≤ 1) if

a) $F(*) = *$

b) $F(A) \vee F(B) \xrightarrow{\sim} F(A \vee B)$

**Excisive codition**

The covariant functor F is 1-excisive if it preserves homotopy pushout squares. Equivalently F takes homotopy pushout squares to homotopy pullback squares. (in this case $\pi_*^s$ has the Mayer-Victoris sequence).

**Example 4.4.** A homotopy functor $F : Top* \to Top*$ is excisive if the covariant functor F takes homotopy pushout to homotopy pullback squares. (If we take homotopy group of the functor F this will have the Mayer- victoris sequence a-rming the excisive condition of the functor F).

**Manifold Calculus**

**Example 4.5.** Contravariant functor $F : \vartheta(\mathbb{R}^n) \to Top$. Where we can think of F to be a functor on the category of open subsets of R*n*. Hence F is is excisive or degree ≤ 1 if we consider the homotopy pushout of this category.

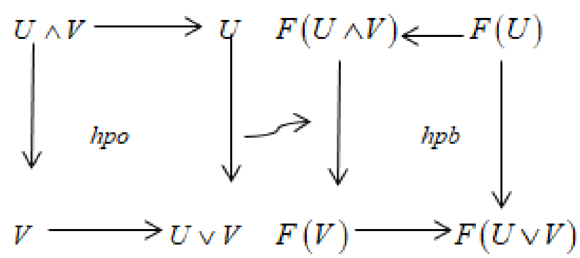


**Figure 8** : excisive diagram of manifold calculus

Hence the contravariant functor F is 1-excisive since it preserves the homotopy pushout squares. Equivalently F takes homotopy pushout squares to homotopy pullback squares.

**5. Constructing Approximation**

**5.1. Approximation via Cross-Effect.**

**Example 5.1** Considering the settings $F : C \to D$ with C being pointed with co product and D as abelian group. $F : C \to D$ Reduced implies $F(0_C) \to 0_D$. The second crosseffect measures the failure of F to be additive. Hence we can define the linear cross-effect of the covariant functor $F : C \to D$ as

$$Cr_2F(A,B) := \ker\left(F(A \vee B) \to FA + FB\right)$$

$$\therefore F(A \vee B) \cong FA + FB + Cr_2F(A,B)$$

Therefore F is additive if $Cr_2F(A,B) = 0$, $\forall A, B \in C$.

**Example 5.2.** Considering $F : Top* \to Sp$. We can define the linear cross-effect of the functor as

$$Cr_2(A,B) = hofiber\left(F(A \vee B) \to FA \vee FB\right)$$

$$\therefore F(A \vee B) \cong FA \vee FB \vee Cr_2F(A,B)$$

Hence F is additive iff $Cr_2F(A,B) = 0$

### 5.2. Approximation via Suspension (To get Excisive Functors)

**Example 5.3**. Considering *F* : *Top*∗ → *Top*∗ reduced homotopy functor. Want to naively force F to be 1-excisive or excisive.
**Note.** The difference between additive functors and excisive functors is that one can take push out squares that don't just have the initial object in this top hand corner.
For any base space X, there is a nice homotopy pushout that takes the form below,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{hpo}"] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd} \quad \sim \quad \begin{tikzcd} FX \arrow[r] \arrow[d] & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

**Figure 9** : Homotopy pushout of excisive functors

Where $CX$ is the cone and $\sum X$ is the suspension (reduced).
And from definition of excisive functors, a functor is excisive if it takes homotopy pushout squares to homotopy pull back squares. Hence if F is excisive then the output of the figure 9 will be a pullback and FX will be equivalent to the pullback of the remaining parts of the square.
Hence FX should be a pullback of the remaining square.
i.e.

$$FX \to T_1F = ho\lim \left[ \begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd} \right]$$

**Figure 10** : Pullback of the remaining square

If F is excisive then $F \xrightarrow{\sim} T_1F$ . But $T_1F$ need not be excisive. However $T_1F$ is closer to being excisive than the original functor *FX*.
If we iterate this construction then we will eventually be arriving at something excisive.
Thus the essence of Goodwillie construction.
Hence $P_1F = ho\lim\left(F \to T_1F \to T_1T_1F \to ...\right)$

$F \to P_1F$ is an excisive approximation.

### 5.3. Higher Degree Polynomial

A continuous function *f* : R → R is quadratic if

$$f(x+y+z) = f(x+y) + f(y+z) + f(z+x) - f(x) - f(y) - f(z) + f(0)$$

### 5.4. Higher Cross effect

We have talked about the second cross-effect being the basic object of additivity.
Hence to talk about the higher versions of additivity we would need higher cross effect to measure the failure of the functor being *n*-additive. From the setting *F* : *C* → *D* where C was pointed with co-product and D being an abelian group. Hence we can define the nth
cross-effect of the functor *F* : *C* → *D* as

$$Cr_nF(A_1...A_n) =$$
$$Cr_nF\left(Cr_{n-1}F\left(A_1...A_{n-2},-\right)A_{n-1},A_n\right)$$

Then $F(A_{1v}...A_n) \simeq \prod_{s=(1,...,n)} Cr_{|s|}F\left(\{A_i\}\right)_{i\in S}$

F is degree ≤ n (n – additive) if $Cr_{n+1}F \simeq 0$

### 6. Conclusion

The study focuses on linear polynomial functors. i.e. the study explained polynomial functors in the algebraic and topological settings with the topological setting focusing on the Goodwillie case, the embedding case and the orthogonal case (thus concentrating on the linear case and generalizing it to the *n*-polynomial case) From the study we looked at approximating a functor via the cross-effect and suspension to get excisive functors.

### REFERENCES

Abbott, M., Altenkirch, T., and Ghani, N. (2003). Categories of containers. In *International Conference on Foundations of Software Science and Computation Structures*, pages 23-38. Springer.
Awodey, S. (2010). *Category theory.* Oxford university press.
Barnes, D. and Oman, P. (2013). Model categories for orthogonal calculus. *Algebraic & Geometric Topology*, 13(2):959-999.
Bisson, T. and Joyal, A. (1995). The dyer-lashof algebra in bordism. *CR Math. Rep. Acad. Sci. Canada*, 17(4):135-140.
Eilenberg, S. and MacLane, S. (1945). General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58(2):231-294.
Ezearn, J. and Obeng-Denteh, W. (2015). Aproximations in divisible groups: Part i. *Physical Science International Journal*, pages 112-118.
Girard, J.-Y. (1988). Normal functors, power series and λ-calculus. *Annals of pure and applied logic*, 37(2):129-177.
Goodwillie, T. G. (1990). Calculus i: The first derivative of pseudoisotopy theory. *K-theory*, 4(1):1 27.
Goodwillie, T. G. (1991). Calculus ii: analytic functors. *K-theory*, 5(4):295-332
Goodwillie, T. G. (2003). Calculus iii: Taylor series. *Geometry & Topology*, 7(2):645-711.
Hazewinkel, M. (Ed.). (2013). *Encyclopaedia of mathematics : Coproduct—Hausdorff—Young inequalities* (Vol. 2). Springer.
Jay, C. B. and Cockett, J. R. B. (1994). Shapely types and shape polymorphism. In *European Symposium on Programming*, pages 302-316. Springer.
Knighten, R. L. (2007). Notes on category theory.
Leinster, T. (2016). Basic category theory. *ArXiv preprint arXiv:1612.09375.*
Macdonald, I. G. (1998). *Symmetric functions and Hall polynomials*. Oxford university press
Moerdijk, I. and Palmgren, E. (2000). Wellfounded trees in categories. *Annals of Pure and Applied Logic*, 104(1-3):189-218.

Nkrumah, F. K., Gyampoh, S. A., and Obeng-Denteh, W. (2019). The universal coeffcient theorem for homology and cohomology: an enigma of computations.

Panangaden, P. (2012). Brief notes on category theory.

Pirashvili, T. (2000). Polynomial functors over finite fields. *Séminaire Bourbaki,* 42:369-388.

Phillips, S., & Wilson, W. H. (2010). Categorial compositionality : A category theory explanation for the systematicity of human cognition. *PLoS computational biology*, *6*(7), e1000858.

Robin, T. (2020). An introduction to category theory.

Setzer, A. and Hancock, P. (2005). Interactive programs and weakly final coalgebras in dependent type theory (extended version). In *Dagstuhl Seminar Proceedings. Schloss* Dagstuhl-Leibniz-Zentrum für Informatik.

Turi, D. (2001). Category theory lecture notes. *University of Edinburgh. URL http://www. dcs. ed. ac. uk/home/dt/CT/categories. pdf*.

Weiss, M. (1995). Orthogonal calculus. *Transactions of the American mathematical society*, 347(10):3743-3796.

Zigli, D. D., Obeng-Denteh, W., and Owusu-Mensah, I. (2017). On the candid appraisal of the proof of õ1 (s, xo) as a fundamental group with respect to ‘o.