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THE FIXED POINT AS A PERIOD-1 RECURRENT IN TOPOLOGICAL DYNAMICAL SYSTEMS

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ABSTRACT
The behavior of the dynamical orbit of a system by describing it relies on the method used. The paper uses the logistic function to illustrate and describe the fixed point of the periodic–like recurrence as a periodic -1 recurrent. The study is based on **Theorem 1:**$(X, T)$is a fixed (Stationary) recurrent point, **iff** for all $x \in X$ and an operator $T: X \to X$ a continuous map and any neighborhood $U \subset X$ then,$\text{U} \coloneqq \{\boldsymbol{n} \in \boldsymbol{R}: \boldsymbol{T^n}(\boldsymbol{U}) \neq \emptyset\} \therefore \boldsymbol{T^n}(\boldsymbol{x}) = \boldsymbol{x} \in \boldsymbol{U}, \boldsymbol{n} \in \boldsymbol{R}, \boldsymbol{n} > \boldsymbol{0}$, **Theorem 2:** a point $\mathbf{x}$ is periodic -1 or fixed point if $\mathbf{F}(\mathbf{x}) = \mathbf{x}$, $\mathbf{Fix}(\mathbf{F}) = \mathbf{Per_1}(\mathbf{F})$ and form a fixed (Stationary) recurrent point $\mathbf{T^n}(\mathbf{x}) = \mathbf{x} \in \mathbf{U}, \mathbf{n} \in \mathbf{R}, \mathbf{n} > \mathbf{0}$ and, **Definition 7:** a point $\boldsymbol{x} \in \boldsymbol{X}$ is said to be recurrent if for any neighborhood $\boldsymbol{U}$ of$\boldsymbol{x}$, there exists an integer $\boldsymbol{n} \geq \mathbf{1}$such that $\boldsymbol{T^n}(\boldsymbol{x}) \in \boldsymbol{U}$ through the application of the logistic function.
The application of the logistic function on the two theorems (**Theorem 1** and **Theorem 2**) and **Definition 7** explained that period-1 recurrent only exists when there is the existence of fixed point (periodic orbits) which depends solely on the initial point and the parameter $\boldsymbol{\alpha}$ of the logistic function..

Keywords: periodic – like, fixed point, recurrent, logistic function, attracting, repelling, period-1, parameter.

INTRODUCTION

Preliminaries

In Dong & Tian (2018), the periodic point which is the simplest type of recurrence is geometric growth that grows faster but does not exist naturally and in irrational rotation, there are no periodic points. Hence recurrence is absent. A system is recurrent when there is a constant state in time called stationary through evolution. This is the simplest type of evolution but beyond it is periodic evolution. These two evolutions that are stationary and periodic are all recurrent. Under these, there is a period which is fixed i.e. fixed point, where the point in the system returns arbitrarily to the same state always and in that there is an exact repetition of its happenings in the formal period.

In the study of dynamical systems, the logistic map is one of the models and it is an interval map. It is an equation that is dependent on a parameter to predict the behavior or the nature of the system. Logistic Map or Function: $T(x) = ax(1 - x)$
Where a is the parameter which lies within 0 and 4 inclusive, i.e.$0 \leq a \leq 4$ or $a \in [0,4]$ and $x \in [0,1]$

The logistic map is a type of map which has a fixed point of 0 irrespective of its manipulations. The outcome of the system may be stable or unstable depending on the parameter. That is at $a < 1$ or $1 < a < 3$ is a stable fixed point system and $a > 1$ or $a > 3$ is also unstable fixed point for $x = 0$ and $x = 1 - \frac{1}{a}$ respectively. Beyond the fixed point is the periodic point and at $a = 3$ is a bifurcation which is called flip bifurcation. The outcome of period-2 of periodic point creates bifurcation and this type of periodic point is called period-doubling bifurcation. When there is instability as a result of an altered parameter, then there is bifurcation. The stability of a system may be either fixed point or period n point. This can be determined by the absolute value of the derivative of the map; that is, $|f'(x_0)| > 1$ the point x_0 is unstable, which means that the point will move away from x_0. If $|f'(x_0)| < 1$ the point is stable, which indicates that the close point moves towards the point x_0. In all at stability, the point attracts the whole neighborhood and for unstable, the points repel in a neighborhood (Oliver, 2005).

Basic Definitions

Definition 1.0: (**Fixed Point**)
Let f be a continuous mapping of [0, 1] into [0, 1], $\exists z \in [0,1]$ such that $f(z) = z$. Hence the point z is called a fixed point.
Moreover, according to Layek (2015), the idea of a fixed point is basically to analyze the local behavior of a system. Hence the fixed point in basic terms is a constant or equilibrium solution of a given system. In other words, it is regarded as the invariant solution or the critical point or a stationary point or an equilibrium point. In the continuous dynamical system, a point in a system is a fixed point of a flow produced by an autonomous system $\dot{x} = f(x), x \in R^n \Leftrightarrow \emptyset(t, x) = x$

NOTE: In the study of a continuous dynamical system, a flow may have no fixed points, exactly one fixed point and, either finite or infinite number of fixed points.
For example**,** the flow $\dot{x} = 10$ has no fixed point, $\dot{x} = x$ has only one fixed point, $\dot{x} = 2x^2 - 8$ has two fixed points and $\dot{x} = \cos x$ has an infinite number of fixed points.

Definition 2.0 (Dynamical System)
Let $T: X \longrightarrow X$, where T is a rule(function) and X is a space, then (T, X) is called the dynamical system, where T is a continuous function.

Definition 3.0 (Iteration)
If f is a function, $f^n(x)$ means $f(f(\ldots f(x) \ldots))$ that is f composed with itself n times, then f is **iterated** n times. **(Wesley, 2003)**

Definition 4.0 (Equilibrium Point)
The period one point which is called the **fixed point** is also called the equilibrium point. That is, if $f: R \longrightarrow R$, **THEN** $f(x) = x$ is the

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points that map to itself after one iteration.
There are two kinds of equilibrium points namely **sink or attractor** and **source or repeller**. The sink or attractor is when neighboring trajectories approach asymptotically to the point and the source or repeller is when neighboring trajectories move away from the point.

Definition 5.0 (Periodic Point)
The **periodic point**: the points that map to themselves after several iterations. Let $f: X \longrightarrow X$, $x_0 \in R$, the point x_0 is a periodic point of period n for f **if** $f^n(x_0) = x_0$

Definition 6.0: Continuous-time and Discrete-time: (Scheinerman, 2000)
In the dynamical system, the time may be continuous or discrete-time.

1. **Continuous-time**: Let $\bar{x}$ be a fixed point of the continuous dynamical system, i.e $x' = f(x)$, then;
   - **a.** If $f'(\bar{x}) < 0$, $\bar{x}$ is a stable fixed point
   - **b.** If $f'(\bar{x}) > 0$, $\bar{x}$ is an unstable fixed point
2. **Discrete-time**: Let $\bar{x}$ be a fixed point of the discrete dynamical system, that is $x_{(n+1)} = f(x_n)$, then;
   - **a.** if $|f'(\bar{x})| < 1$, then $\bar{x}$ is a stable fixed point
   - **b.** If $|f'(\bar{x})| > 1$, then $\bar{x}$ is an unstable fixed point

Note: A fixed point is called stable if it tends to a particularly critical point

MAIN THEOREMS AND DEFINITION

The Fixed Recurrent Point Using Logistic Function
The first periodic point (fixed point) shows recurrent formation. The formation of this behavior (recurrent) is a result of different values for the control parameter $\boldsymbol{\alpha}$. The two behaviors of a ***fixed point***, '***attracting*** and ***repelling***' are all ***recurrent***.

Definition 7.0: If **(X, T)** is a dynamical system, a point $\boldsymbol{x} \in \boldsymbol{X}$ is said to be recurrent if for any neighborhood $\boldsymbol{U}$ of $\boldsymbol{x}$, there exists an integer $\boldsymbol{n} \geq \mathbf{1}$ such that $\boldsymbol{T^n}(\boldsymbol{x}) \in \boldsymbol{U}$

Theorem 1.0: A compact topological dynamical system (X, T) is a fixed (Stationary) recurrent point, **iff** for all $x \in X$ and an operator $T: X \to X$ a continuous map and any neighborhood $U \subset X$ then,

$$\bigcup := \{\boldsymbol{n} \in \boldsymbol{R}: \boldsymbol{T^n}(\boldsymbol{U}) \neq \emptyset\}$$

$$\therefore\ \boldsymbol{T^n}(\boldsymbol{x}) = \boldsymbol{x} \in \boldsymbol{U}, \boldsymbol{n} \in \boldsymbol{R}, \boldsymbol{n} > \mathbf{0}$$

PROOF: Given $T: X \to X$, $for\ x \in X$ where X is a compact metric space and T is continuous or homeomorphism. Taking a neighborhood U in X a non-empty;
That is, $\mathbf{U} \subset \boldsymbol{X} \implies \boldsymbol{T}(\boldsymbol{U}) \subset \mathrm{U}$
Let $\boldsymbol{X_0}$ be a Set and Let $\boldsymbol{x} \in \boldsymbol{X_0}$.
THEN, $\boldsymbol{X_1} = \{\boldsymbol{T^n}(\boldsymbol{x}), \boldsymbol{n} = \mathbf{1}, \mathbf{2}, \ldots\}$ is within the neighborhood $\boldsymbol{U}$ contained in $\boldsymbol{X_0}$
Implying $\boldsymbol{X_1} = \boldsymbol{X_0}$ is a minimal set. Therefore, $\boldsymbol{x} \in \{\boldsymbol{T^n}(\boldsymbol{x}), \boldsymbol{n} > \mathbf{0}\}$
Hence $\boldsymbol{x}$ *is a* ***fixed recurrent point (period-1 recurrent)***.
□

Example 1.0: Given $f: X \to X$ as a function which is homeomorphism in nature, defined by $f(x) = \alpha x$, $\forall\, x \in X, \alpha \in [0,1]$, called the control parameter.

PROOF: At a fixed, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$, $\boldsymbol{Fix}(\boldsymbol{F}) = \boldsymbol{Per_1}(\boldsymbol{F})$
Then, $\boldsymbol{\alpha x} = \boldsymbol{x} \Longrightarrow \boldsymbol{\alpha x} - \boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}(\boldsymbol{\alpha} - \mathbf{1}) = \mathbf{0} \Longrightarrow \boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{\alpha} - \mathbf{1} = \mathbf{0} \Longrightarrow \boldsymbol{\alpha} = \mathbf{1}$
$\therefore \boldsymbol{Fix}\ (\boldsymbol{F}) = \mathbf{0}$, Then the set of a fixed point is determined by $\boldsymbol{Fix}(\boldsymbol{F}) = \boldsymbol{R}$.

NOTE: The fixed recurrent point as demonstrated by **(Mensah et al, 2016)** is a situation where a function under operation with an initial value returns infinitely to the same value. This then gives orbits/trajectory that converges and returns to a **constant or specific** value.

Remarks: The outcome of **Example 1.0** affirms the assertion that at a fixed recurrent point a function is continuous, stable and, equilibrium irrespective of the number of iterations. This is possible for a specific range or the initial that the control parameter carries.

Example 2.0: Let $f: X \to X$ be continuous, where $x \in [0,1]$, defined by $f(x) = \alpha(x - x^2)$, $\forall\ \alpha \in [1,4]$, then for a fixed recurrent point $\alpha < 3$, the function converges. But for $\alpha > 3$, we have the fixed recurrent point which repels **(Mensah et al, 2016)**.

Theorem 2.0: Dynamically, a point $\mathbf{x}$ is periodic -1 or fixed point if $\mathbf{F}(\mathbf{x}) = \mathbf{x}$, $\mathbf{Fix}(\mathbf{F}) = \mathbf{Per_1}(\mathbf{F})$ and form a fixed (Stationary) recurrent point $\mathbf{T^n}(\mathbf{x}) = \mathbf{x} \in \mathbf{U}, \mathbf{n} \in \mathbf{R}, \mathbf{n} > \mathbf{0}$.

RESULTS AND DISCUSSION

Using the logistic function to Illustrate theorem 2.0 as a periodic-1 recurrent

Illustration (Theorem 2.0): Using the logistic function, $\boldsymbol{x} \in [\mathbf{0}, \mathbf{1}]$, defined by $(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{x} - \boldsymbol{x^2})$, $\forall\ \boldsymbol{\alpha} \in [\mathbf{1}, \mathbf{4}]$.
So, at $\alpha = 1.1$ implies $x_0 = \frac{\alpha - 1}{\alpha} = \frac{1}{11} = 0.090909$, the iteration of the function $f(x) = \alpha(x - x^2) = 0.09090909091$ a recurring decimal which is also the same as the initial value irrespective of the number of iteration.

Table 1: **Iteration of** $\boldsymbol{f}(\boldsymbol{X}) = \boldsymbol{\alpha}(\boldsymbol{x_n} - \boldsymbol{x^2}_{\boldsymbol{n}})$ with $\alpha = 1.1$ and $x_0 = 0.090909$

n	0	1	2	3	4
x_n	0.090900	0.090901	0.090902	0.090903	0.090904

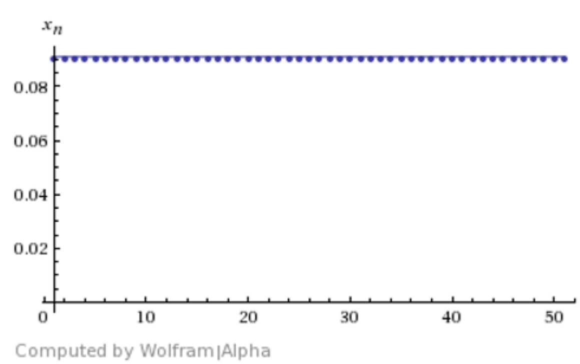

Figure 1: The equilibrium nature of $\mathrm{f(X)} = \alpha(\mathrm{x_n} - \mathrm{x^2}_\mathrm{n})$ $(\alpha = 1.1, \mathrm{x_0} = 0.090909)$

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Table 1 and **Figure 1** give a clear picture of a periodic – 1 orbit which exhibits the nature of a recurrent as stated by **Theorem 1, Theorem 2,** and, **Definition 7.0** having the same value after several iterations. The value of α **(**control parameter**)** when substituted into $x_0 = \frac{\alpha-1}{\alpha}$ when using the logistic function produced trajectories that give exact/same outcome as (the initial value) x_0 **fixed** after several manipulations. Therefore, it is evident that x as a fixed point of periodic-like recurrent forms a periodic -1 recurrent point.

Since a periodic -1/fixed point $F(x) = x$, and $Fix(F) = Per_1(F)$ is a fixed (Stationary) recurrent point $T^n(x) = x \in U, n \in R, n > 0$, **Theorem 2** is illustrated.
The outcomes of **Table 1** and **Figure 1** again show that a fixed point that is recurrent is; '**continues**, **equilibrium** and **stable**'.
Proposition 1: If the fixed point $\mathbf{x}$ is a recurrent point then $\mathbf{x}$ is either **attracting** or **repelling.** In others words **stable** and **unstable**

Proof: By considering the logistic function; $f(x) = \alpha(x - x^2)$, $\forall\ 0 < x < 1$ *and* $\alpha \in [1, 4]$.We show by finding the fixed points and determining the **stableness** and **unstableness** of the fixed point.
For the fixed points; $f(x) = \alpha(x - x^2)$ *implies* $x = 0$ *and* $x = \frac{\alpha-1}{\alpha}$. Where the stability is dependent on the multiplier$f'(x) = \alpha - 2\alpha x$;
At $x = 0$ $f'(0) = \alpha$ implies $\alpha < 1$ is **stable** and $\alpha > 1$ is **unstable** for the **origin** $(x = 0)$**.** Hence, all α the fixed point stays at the origin, which then means that the origin $x = 0$ is the fixed point irrespective of α**.**

At $x = \frac{\alpha-1}{\alpha}$, $f'\left(\frac{\alpha-1}{\alpha}\right) = -\alpha + 2$ *implies* $x = \frac{\alpha-1}{\alpha}$ is **stable** for $1 < \alpha < 3$ and **unstable** for $\alpha > 3$, and if $\alpha \geq 1$,$x = \frac{\alpha-1}{\alpha}$ is in the range of x

Therefore, since it is true for fixed points to be stable and unstable, then recurrent fixed points are stable and unstable. □

Illustration: By using the logistic function; $(x) = \alpha(x - x^2)$, $\forall\ \alpha \in [1, 4]$.

Attracting

Table 2: The Stability table of the Periodic Nature

Period	Iterates	Linear stability
1	0.	Unstable
1	0.0909091	Stable

Example 3: Graphical and Tabula Illustration of **attracting/stable fixed point** as a **period- 1 recurrent** formation using the logistic function when $\alpha < 3$ and where the fixed point which serves as the initial value x_0 is not dependent on α. Let $\alpha = 2.5 < 3$ and $x_0 = 0.20$

$f(X) = \alpha(X - X^2) = 2.5(x_n - x^2{}_n)$, at $x_0 = 0.20$

Table 3: **Iteration of** $f(X) = 2.5(x_n - x^2{}_n)$, at $x_0 = 0.20$,

n	0	1	2	3	4
x_n	0.20000	0.40000	0.60000	0.60000	0.60000

As indicated in **Table 3** where $\alpha = 2.5 < 3$ that is taking a control parameter $\alpha < 3$ and $x\epsilon[0, 1]$ the orbits or sequence **converges** after successive iterations and then return to one specific value. The values **0.20000** and **0.40000** forms **transient** which when dumped/cut off, forms **recurrence right** from **…,0.60,…** of the **orbits or** the **trajectory** hence **(period-1 recurrent)** since the third value keeps repeating over and over again irrespective of the number of iterations with the exemption of the first two values; that is **{…, 0.60, 0.60, 0.60, …}**

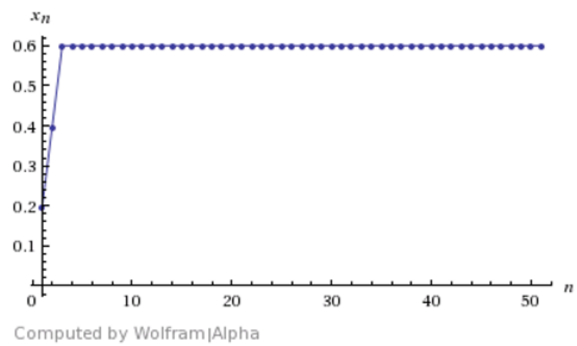

Figure 2: Graph of period-1 uniformity of $f(X) = \alpha(x_n - x^2{}_n)$ with $\alpha = 2.5$ and $x_0 = 0.2000$

Table 4: The Stability of Periodic Nature

Period	Iterates	Linear stability
1	0.	Unstable
1	0.60	Stable

Figure 2 and **Table 4** show or give an outcome that is continuous, stable, and equilibrium of the function after several iterations as it returns to a specific value. Therefore the point that **converges** forms the **period-1 recurrent**.

Repelling

Example 4: using the logistic function to illustration/show the **repelling fixed point** as a **period-1 recurrent or not** when $\alpha > 3$ and$x\epsilon[0,1]$.

Let $\alpha = 3.1 > 3$ then $x_0 = \frac{3.1-1}{3.1} = \frac{21}{31} = 0.677$ is the **initial point** and the **fixed point**.

$f(X) = \alpha(X - X^2) = 3.1(x_n - x^2{}_n)$
Now we iterate the function with a different initial value and then check what happens.
$x_0 = 0.10000$

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Table 5: Iteration of $\boldsymbol{f(X) = 3.1(x_n - x^2{}_n)}$, at $x_0 = 0.10000$

n	0	1	2	3	4
x_n	0.10000	0.27900	0.62359	0.72765	0.61435

The outcome when $\boldsymbol{x_0 = 0.10000}$, as shown by **Table 5**, gives an **unstable** condition. That is after iterating the function successively, it stays far from **0.677** that is moving away from the fixed point, and then forms trajectories that are repelling. These orbits or trajectories **{0.10, 0.27900, 0.62359, 0.72765, 0.61435, …}** disobey the rules of stableness hence showing no formation of the **period – 1 recurrent** of the fixed point but gives **different** outcomes on both its left and right.

However, when $x_0 = 0.677$ which serves as the fixed point is used as the initial value there is a formation of the **period – 1** recurrent after approximating the iterated values obtained through the function. **Table 6** and **Figure 3** below show that the resulting values after iterating the function tend to move away or diverge gradually on both the left and right of the fixed point.
Therefore, at $x_0 = 0.677$

Table 6: **Iteration of** $\boldsymbol{f(x) = 3.1(x_n - x^2{}_n)}$, at $x_0 = 0.67700$

n	0	1	2	3	4
x_n	0.67700	0.67788	0.67691	0.67798	0.67681

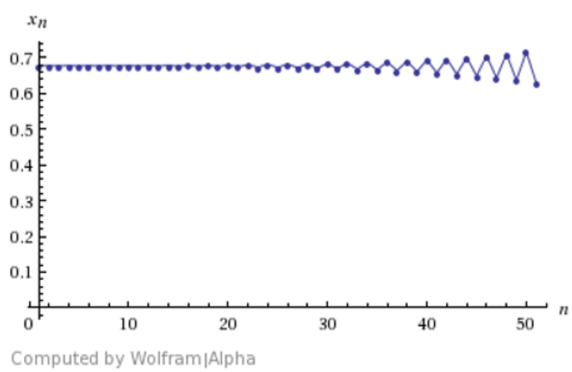

Figure 3: Graph of $(X) = 3.1(x_n - x^2{}_n)$, $(x_0 = 0.67700)$

Therefore, when the parameter is greater than 3 ($\boldsymbol{\alpha > 3}$) there is **unstableness** for **period-1recurrent** of the periodic – like recurrence as shown in **Figure 3** above and **Table 7** below

Table 7: The Stability table of the Periodic Nature

Period	Iterates	Linear stability
1	0.	Unstable
1	0.677419	Unstable
2	0.558014,0.764567	Stable

Table 7, Figure 3 and **Table 8** shows that when the parameter $\boldsymbol{\alpha}$ is greater or higher than **3** that is $\boldsymbol{\alpha}$**=3.5 > 3** where the fixed point is $\boldsymbol{x_0 = 0.71}$ the out of the function diverges totally from the **fixed point** showing no **recurrent formation** making it **unstable.**
Also at $\boldsymbol{x_0}$ **= 0.71**

Table 8: Iteration of $f(X) = 3.5(x_n - x^2{}_n)$, at $x_0 = 0.7100$,

n	0	1	2	3	4
x_n	0.7100	0.72065	0.70460	0.72849	0.69227

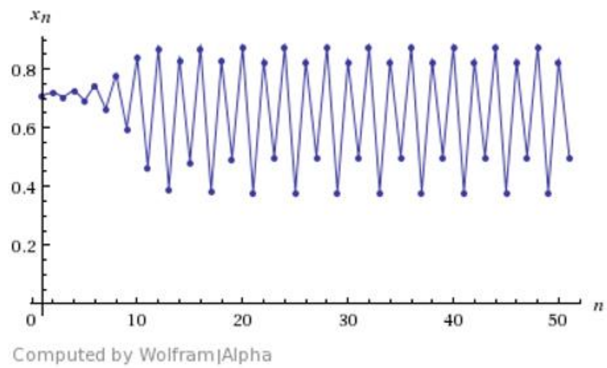

Figure 4: The graph of the repelling point of $f(X) = 3.5(x_n - x^2{}_n)$ $(x_0 = 0.7100)$

Table 9: Stability table for the Periodic Nature

Period	Iterates	Linear stability
1	0.	Unstable
1	0.0714286	Unstable
2	0.428571,0.857143	Unstable
4	0.38282,0.856941,0.500884,0.874997	Stable

Finally, from the illustrations, it has been shown that within the intervals $-\mathbf{1} < \boldsymbol{\alpha} < \mathbf{1}$ and $< \mathbf{3}$, there is a formation of a type of periodic – like recurrence called **period – 1 recurrent** of the logistic function of the fixed point which is **attracting** in nature.
However, the formation of this **period – 1 recurrent** depends largely on the initial point and control parameter $\boldsymbol{\alpha \epsilon (0, 1) or\ (3, 4)}$, and at these intervals of the parameter $\boldsymbol{\alpha}$ the fixed point of the logistic function **repels** hence the formation of this **period – 1 recurrent** of the periodic – like recurrence is not assured.

At a frequent operating of the system, the account of this $\mathbf{x} \in \mathbf{X}$ keeps repeating at a fixed irrespective of the parameter it carries or attached to it. **Berkhoff** termed such a situation as a **stronger version of recurrence** hence called it **uniform recurrence**. This **uniform recurrence** is what we **normally refer** to as the **neutral fixed point (ISSAKA. I et al, 2018),** and notwithstanding, it is neither repelling nor attracting to a constant point.

Conclusion

The study shows that in a dynamical system the periodic orbits of a system give rise to a recurrence formation if it infinitely returns to the initial point after several iterations. The study also shows that the fixed point as a period-1 recurrent in a topological dynamical system or dynamical system is a type of recurrence formation. The application of the logistic function on the two theorems (**Theorem 1** and **Theorem 2**) and **Definition 7** explained that period-1 recurrent only exists when there is the existence of fixed point (periodic orbits) which depends solely on the initial point and the parameter $\boldsymbol{\alpha}$ of the logistic function. Mainly on the intervals $-\mathbf{1} < \boldsymbol{\alpha} < \mathbf{1}$ and $< \mathbf{3}$ for **attracting** and $\boldsymbol{\alpha\epsilon}(\mathbf{0},\mathbf{1}) or\ (\mathbf{3},\mathbf{4})$ for **repelling**.

Therefore the **effect of the period – 1 recurrent** of the periodic – like recurrence as in the **topological dynamical system** is that; it creates a regime we called the **stability or stableness**, thereby, showing the uniformity of the space regardless of its subsequence operations.

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