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MATHEMATICAL ANNUITY MODELS APPLICATION IN CASH FLOW ANALYSIS

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ABTRACT

Many problems arising in the mathematics of finance involve identical money flows at regular time intervals and are solved by appropriate valuation at a focal date or by setting up an equation of value. This paper shows how such problems can be viewed as special cases of a certain class of first-order difference equations. To illustrate the annuity models, we draw tables to compare the reinvestment of benefit from a fixed deposit and the usual simple interest method. We also compare the cost efficiency between Amortisation and Sinking fund loan repayment as prevalent in financial institutions.

Keywords: Annuity, Amortisation, Sinking Fund, Present and Future Value Annuity, Maturity date and Redemption value.

INTRODUCTION

The term annuity refers to a series of equal payments made at equal intervals of time arising from problems in the mathematics of finance and long-term maturity loans. An annuity is a sequence of payments made over uniform time intervals and may be ordinary (paid at the end of the interval) or due (paid at the beginning of the interval).

Common examples include interest payments on bonds, computation of interest in bank accounts, loan repayments such as amortisation (capital recovery) and Sinking fund (Stelson 1967). The problems can take the forms such as calculating:

i the future value of annuity
ii the present value of an annuity
iii the instalment loan repayment
iv the price of a bond given the par value, maturity date, redemption value and the coupon rate (interest rate)

The objective of this paper is to provide a unified general formulation to handle problems that include ordinary annuities as special cases. The formulation is based on difference equation approach.

A difference-equation approach to problems in finance is not new. Both Kotiah (1991) and Pandey (1999) have discussed the computation of the payments of an instalment loan (amortisation) and bonds in form of sinking fund, given the amount, the interest rate, and maturity date and redemption value

GENERAL FORMULATION

Consider a fund with an initial balance P where P>0. At the end of a period t (a period may be a month, a quarter, half a year, a year) a payment q_t is made. If the payment increases the balance, $q_t > 0$, if it decreases the balance $q_t < 0$ and $q_t = 0$ if no payment is made. Suppose the balance bears an interest r at the end of each period. If nominal yearly rate is given, then r equals nominal rate divided by the number of compounding or payment periods in one year. We shall now find the balance of the fund after n periods.

Consider X_t to be the balance of the fund after t periods, then for r>0

$$X_{t+1} = X_t(1+r) + q_{t+1},\ t = 0, 1, 2, \ldots, n-1 \quad \ldots \quad (1)$$

$$X_o = P \quad \ldots \quad (2)$$

The solution to equation (1) subject to (2) with assumption that $X_t > 0$, for t = 0, 1, 2, …, n is found by successively setting, t = 0, 1, … n – 1, in equation (1). That is for

$X_2 = X_1(1 + r) + q_2$

Since $X_1 = X_0(1 + r) + q_1 = P(1 + r) + q_1$

Then $X_2 = (P(1 + r) + q_1)(1 + r) + q_2$

$X_2 = P(1 + r)^2 + q_1(1 + r) + q_2$

Then for t = n

$$X_n = P(1+r)^n + \sum_{t=1}^{n} q_t \left(1+r\right)^{n-t}) \ldots \quad (3)$$

ANNUITY

The annuity as defined earlier can be obtained from equation (3) by setting $q_t = C$, where C is constant payment. That is

$$X_n = P(1+r)^n + C\sum_{t=1}^{n}(1+r)^{n-1} \ldots \quad (4)$$

Then by geometric summation the second term in the equation reduces to

$$\frac{C\left\{\left(1+r\right)^n - 1\right\}}{r}$$

Consequently, for $S = X_n$ equation (4) becomes

$$S = P\left(1+r\right)^n + \frac{C\left\{\left(1+r\right)^n - 1\right\}}{r} \ldots \quad (5)$$

ORDINARY ANNUITY

Equation (4) is an annuity with an initial balance of P, if the initial balance is zero, that is P = 0 the equation becomes

$$S = \frac{C\left\{\left(1+r\right)^n - 1\right\}}{r} \quad \ldots \quad (6)$$

where C is a series of payments. For C = 1 the formula is conventionally written as

$$S_n = \frac{\left\{(1+r)^n - 1\right\}}{r} \quad ...(7)$$

which is the unit payment annuity factor.

In investment appraisals, cash flows may grow or decline at either compound or linear rates. A closed form solution can be found for such cash flow (Kotiah 1991; Francis 1995).

LINEAR GROWTH ANNUITY

This supposes that the annuity be expressible in linear equation for that is

$q_t = C + (t-1)\,b$, t =1, 2,…, n ...(8)

and setting P = 0 in equation (4)

$$X_n = C\sum_{t=1}^{n}(1+r)^{n-t} + b\sum_{t=1}^{n}(t-1)(1+r)^{n-t} \quad ...(9)$$

Substituting equation (7) in equation (9)

$$X_n = CS_n + +b\sum_{t=1}^{n}(t-1)(1+r)^{n-t} \quad ...(10)$$

We denote the summation term in equation (10) by T and simplifying, we have that

$$T = (1+r)^{n-2} + 2(1+r)^{n-3} + ... + (n-2)(1+r) + (n-1) \quad ...(11)$$

And by multiplying both sides by (1 + r) we obtain

$$(1+r)T = (1+r)^{n-1} + 2(1+r)^{n-2} + ...(n-2)(1+r)^2 + (n-1)(1+r) \quad ...(12)$$

Subtracting equation (11) from equation (12), and using geometric sum, we obtain

$$Tr = S_n - n$$

$$T = \frac{S_n - n}{r} \quad ...(13)$$

Substituting equation (13) into equation (10), we obtain

$$S = CS_n + b\left(\frac{S_n - n}{r}\right) \quad ...(14)$$

Equation (14) is the terminal value of payment that varies in a linear form.

EXPONENTIAL GROWTH ANNUITY

If the payment form a geometric sequence, such that

$$q_t = cb^{t-1}\ , \quad t = 1, 2 ...n,\ b>0,\ b \neq 1 \quad ...(15)$$

And we assume P = 0, equation (3) is given as

$$X_n = C\sum_{t=1}^{n} b^{t-1}(1+r)^{n-t}$$

By factorizing out $(1 + r)^{n-1}$

$$X_n = C(1+r)^{n-1}\sum_{t=1}^{n}\frac{b^{t-1}}{(1+r)}$$

$$S = C(1+r)^{n-1}\left\{1 - \tfrac{b}{(1+r)^n}\right\} / \left(1 - \frac{b}{(1+r)}\right) \quad ...(16)$$

This is the terminal value of compound growth payments.

ILLUSTRATION

To illustrate the method for compounding cash flows, we considered ₦1,000,000.00 fixed deposits at 15% per annum that generates an annuity of ₦ 12, 500.00 at the end of each month as shown in Table 1, (column 2). The reinvested interest at 3% per annum earns an interest of N31.25 at the end of the second month, ₦ 62.50 at the end of the third month. The interest for the second and third month is converted to principal at the end of each quarter of a year, which gives ₦ 37593.75 at the end of the first quarter. The annuity amount to ₦ 308, 764.10, which is ₦ 8764.10 more than the usual simple interest method of ₦ 300, 000.00.

PRESENT VALUE CONCEPT

We have so far shown how compounding techniques are used for the adjustment, of the time value of money. It increases an investor's analytical power to compare cash flows that are separated by more than one period, given the interest rate per period. However it is common practice to translate future cash flow to their present values. The future value of a cash flow is the amount of current cash that is of the equivalent value to the decision maker. The process of determining the present value of future cash flow is called discounting and is obtained by multiplying the future cash flow by $(1+r)^{-n}$

AMORTIZATION

An amortized annuity is a method of repaying a loan. This consists of regular annuity (ordinary and setting) which each payment accounts for both repayment of capital and interest. The debt is said to be amortized if this method is used. Consider equation (4) where the amount S is expected to decay to zero at the focal date.

Thus, $P(1+r)n + C\dfrac{(1+r)^n - 1}{r} = 0$

Solving for C, we obtain

$$C = \frac{-P\left\{r(1+r)^n\right\}}{(1+r)^n - 1} \quad ...(17)$$

C is the fixed payment at the end of each period. P is the initial balance of the fund; R is the interest rate and N is the number of time periods.

The negative sign is indicative of the fact that the balance is reduced with each payment

SINKING FUND ANNUITY

When a loan is collected it attracts interest at the appropriate borrowing rate. If no intermediate payments are made, at the end of the period, the capital plus the interest must be repaid as a lump sum.

The periodic uniform payments c required to accumulate to the required sum to pay the loan at the focal date is called a sinking fund annuity (Francis 1995). Suppose the loan is collected at a proportional borrowing rate i, and grows to the amount S at the focal date.

$$S = P\left(1+i\right)^n \qquad \text{. . . (18)}$$

The constant periodic payment required to accumulate to the required amount S at a discount rate r at the focal date is given by

$$S = \frac{C\left(i+r\right)^n}{r} \qquad \text{. . . (19)}$$

Comparing equations (18) and (19)

$$\frac{C\left(\left(1+r\right)^{n-1}-1\right)}{r} = P\left(1+i\right)^n$$

and solving for C

$$C = \frac{rP\left(1+i\right)^n}{\left(1+r\right)^{n-1}-1} \qquad \text{. . . (20)}$$

Notice that since the borrowing rate is always greater than the investment rate; i > r

So that the ratio

$$\frac{rP\left(1+i\right)^n}{\left(1+r\right)^{n-1}-1}$$

TABLE I. FUTURE VALUE OF ANNUITY GENERATED BY A FIXED DEPOSIT OF ₦ 1, 000,000.00 AT 15% PER ANNUM FOR N MONTHS.

Month	Interest from fixed deposit month end	Interest earned at 3% P. A.	Quarterly Interest Conversion	Amount in fund
1.	12,500	0		
2.	12,500	31.25		
3.	12,500	62.50	93.75	37,593.75
4.	12,500	93.98		
5.	12,500	125.23		
6.	12,500	156.48	375.70	75,469.45
7.	12,500	188.67		
8.	12,500	219.92		
9.	12,500	251.17	659.77	113,629.22
10.	12,500	284.07		
11.	12,500	315.32		
12.	12,500	346.57	945.97	152075.20
13.	12,500	380.19		
14.	12,500	411.84		
15.	12,500	442.69	1234.31	190809.51
16.	12,500	477.03		
17.	12,500	508.27		
18.	12,500	539.53	1524.22	229834.33
19.	12,500	574.59		
20.	12,500	605.84		
21.	12,500	637.09	1817.51	269151.84
22.	12,500	672.88		
23.	12,500	704.13		

To make these models more intuitive Tables 2 and 3 are presented to illustrate the processes of discounting or the concept of present value. Analysis of table 2 shows that an annuity of ₦ 334379.71 per year is needed to pay off the end of ₦ 1, 000,000.00 at the bond of five years. Each payment accounts for both repayment of capital and interest, until the loan is reduce to zero.

For the sinking fund annuity schedule the loan (₦1, 000,000.00) is allowed to compound at 20% per annual to the terminal value at the focal date and a parallel annuity of ₦ 369.056.58 at 15% per annum is set to mature to the terminal value the loan at the focal date (Table 3).

It can be observed that the payment in an amortised loan is less than the payment in a sinking fund loan. This is so because the interest in an amortised loan is reducing while that of sinking fund is increasing. The instalment payment for amortised loan is ₦334.379.71.The amount for sinking fund is ₦ 369.056.56, which is indicate that the borrowing rate is greater than the investment rate. Hence an amortised model is less expensive than using sinking fund model

TABLE 2. AMORTIZATION ANNUITY SCHEDULE OF ₦ 1, 000,000.00 AT 20% PER ANNUM FOR 5 YEARS.

Year	Outstanding debt (year start) (₦)	Payment Made (₦)	Interest Paid (₦)	Principal Repaid (₦)
1	1,000,000.00	334,379.71	-	
2	865,620.29	334,379.71	200,000.00	134,379.71
3	704,364.60	334,379.71	173,124.06	161,255.65
4	510,857.89	334,379.71	140,972.93	193,506.78
5	278,649.76	334,379.71	102,171.58	232,208.13
6	00.00	334,379.71	55,729.95	278,649.76

TABLE 3. SINKING FUND ANNUITY SCHEDULE OF ₦ 1, 000,000.00 FOR 5 YEARS

Year	Outstanding debt (year start) (₦)	Interest paid at 20% P. A	Payment Made (₦)	Interest earned at 15% P. A (₦)	Principal Repaid (₦)
1	1,000,000.00	-	-	-	-
2	1,200,000.00	200,000.00	369,056.56	-	369,056.56
3	1,440,000.00	240,000.00	369,056.56	55,358.48	793,471.60
4	1728,000.00	288,000.00	369,056.56	119,020.74	1,281,548.90
5	2,073,600.00	345,600.00	369,056.56	192,232.34	1,842,837.80
6	2,488320.00	414,720.00	369,056.56	276,425.67	2,488,320.03

CONCLUSION

The difference equation formulation presented here offers another method of treatment of problems relating to discrete cash flows. Such formulations brings to the mathematics of finance the benefits of a well developed range of techniques in applied mathematics as shown in Tables 1-3. The work illustrates how annuity is used to settle both repayment of capital and interest, until the loan is reduce to zero. Also on how the sinking fund annuity is set to mature to the terminal value the loan at the focal date.

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