FULL LENGTH RESEARCH ARTICLE

ON SOME PROPERTIES OF THE BLOCK LINEAR MULTI-STEP METHODS

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ABSTRACT
The convergence, stability and order of Block linear Multistep methods have been determined in the past based on individual members of the block. In this paper, methods are proposed to examine the properties of the entire block. Some Block Linear Multistep methods have been considered, their convergence, stability and order have been determined using these approaches.

Keywords: Block Linear multistep method, Region of absolute stability, Convergence

INTRODUCTION
Continuous integration algorithms for the numerical solution of the initial value problem (IVP) of the form

$$y'(x) = f(x,y), \quad a < x < b, y(a) = y_0 \quad \ldots 1.0$$

has been the area of focus in recent times. The block linear multistep method was developed by Onumanyi, *et al.* (1994), Onumanyi, Sirisena & Jatau (1999) through continuous interpolants based on the work of Lie & Norsett (1986).

The block methods together with their hybrid forms were studied (Onumanyi, *et al.* 2001) and further investigated (Chollom & Onumanyi, 2004; Dauda, *et al.* 2005) and found to be useful for the direct solution of initial and boundary value problems.

The advantages of these methods include (i) overcoming the issue of overlap of pieces of solutions usually associated with the multistep finite difference methods and (ii) they are self starting thus eliminating the use of other methods to obtain starting solutions. Before now, the determination of the order of these block methods, their convergence and plotting their absolute stability regions have been done for the single members of the block and whose result may not be assumed for the entire block. In this paper, we consider some of the properties of the entire block.

Convergence of the block linear multistep method: In the discreet case, a linear multistep method is said to be convergent if for all initial value problem (1.0)

$$y(x) - y_n(x) \to 0 \quad \text{as } h \to 0, x \in [x_0, x].$$

Previously, the convergence of these methods is done considering only the individual members of the block. This paper determines the convergence of the entire block using the Fatunla (1994) approach where each block integrator is represented as a single step block, r-point multi-step method of the form:

$$Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \cdot \\ \cdot \\ y_{n+r} \end{pmatrix}, \quad Y_{m-1} = \begin{pmatrix} y_{n-r} \\ \cdot \\ \cdot \\ y_{n-1} \\ y_n \end{pmatrix}, \quad F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \cdot \\ \cdot \\ f_{n+r} \end{pmatrix}, \quad F_{m-1} = \begin{pmatrix} f_{n-r} \\ \cdot \\ \cdot \\ f_{n-1} \\ f_n \end{pmatrix} \quad \ldots 3.0$$

Definition (1.0) for $n = mr$, for some integer $m \geq 0$ the block method (2.0) is said to be zero stable if the roots

$$\rho(R) = \det\left\{\sum_{i=0}^{k} A^{*i} R^{k-i}\right\} = 0 \quad \ldots 4.0$$

$R_j, J = 1(1)k$ of the first characteristic polynomial satisfies $|R_j| \leq 1$ and for the roots with $|R_j| \leq 1$ the multiplicity must not exceed two

ORDER OF THE BLOCK LINEAR MULTI-STEP METHOD: The linear multi-step method is said to be of order p if $C_0 = C_1 = C_2 = \ldots C_p = 0, C_{p+1} \neq 0$. This approach is normally used to determine the order of the individual members of the block. We extend this approach to determine the order of the entire block. To achieve this, the block linear multi-step method is expressed in the form

$$\sum_{i=0}^{k} \alpha_{ij} y_{n+j=h} \sum_{i=0}^{k} \beta_{ij} f_{n+j} \quad \ldots 5.0$$

where $j = 0, 1, \ldots k$ is a positive integer. Equation (5) is expanded to give the following system of equation.

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} & . & . & . & \alpha_{k1} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} & . & . & . & \alpha_{k2} \\ \alpha_{03} & \alpha_{13} & \alpha_{23} & . & . & . & \alpha_{k3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{01} & \alpha_{1k} & \alpha_{2k} & . & . & . & \alpha_{kk} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ . \\ . \\ . \\ y_{n+k} \end{pmatrix} = h \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{21} & . & . & . & \beta_{k1} \\ \beta_{01} & \beta_{12} & \beta_{22} & . & . & . & \beta_{k1} \\ \beta_{01} & \beta_{13} & \beta_{23} & . & . & . & \beta_{k1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \beta_{01} & \beta_{1k} & \beta_{2k} & . & . & . & \beta_{kk} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ . \\ . \\ . \\ f_{n+k} \end{pmatrix} \qquad ...6.0$$

The expression (6.0) is equivalent to $\sum_{i=0}^{k} \alpha_{ij} y_{n+j=h} \sum_{i=0}^{k} \beta_{ij} f_{n+j}$

where $\overrightarrow{\alpha_0} = \begin{pmatrix} 01 \\ 02 \\ 03 \\ . \\ . \\ . \\ 0k \end{pmatrix}, \overrightarrow{\alpha_1} = \begin{pmatrix} 11 \\ 12 \\ 13 \\ . \\ . \\ . \\ 1k \end{pmatrix}, ...\overrightarrow{\alpha_k} = \begin{pmatrix} k1 \\ k2 \\ k3 \\ . \\ . \\ . \\ kk \end{pmatrix}$ and $\overrightarrow{\beta_0} = \begin{pmatrix} 01 \\ 02 \\ 03 \\ . \\ . \\ . \\ 0k \end{pmatrix}, \overrightarrow{\beta_1} = \begin{pmatrix} 11 \\ 12 \\ 13 \\ . \\ . \\ . \\ 1k \end{pmatrix}, ...\overrightarrow{\beta_k} = \begin{pmatrix} k1 \\ k2 \\ k3 \\ . \\ . \\ . \\ kk \end{pmatrix}$

Adopting the order procedure used in the single case for the block method, we recall that

$$\ell_h y(x) = \sum_{i=0}^{k} \left[\overrightarrow{\alpha_j} y(x + jh) - h \overrightarrow{\beta_j} f(x + jh, y(x + jh)) - \right] \qquad ...7.0$$

where $y(x)$ is the exact solution satisfying $y' = f(x, y(x))$. Carrying a Taylor series expansion of (7.0) about x yields the equation

$$\ell_h y(x) = \overrightarrow{C_0} y(x) + \overrightarrow{C_1} h y'(x) + \overrightarrow{C_2} h^2 y''(x) + ... + \overrightarrow{C_q} h^q y^q(x) \qquad ...8.0$$

where $\overrightarrow{C_0} = \begin{pmatrix} c_{01} \\ c_{02} \\ c_{03} \\ . \\ . \\ . \\ c_{0p} \end{pmatrix}, \overrightarrow{C_1} = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ . \\ . \\ . \\ c_{1p} \end{pmatrix} ...\overrightarrow{C_p} = \begin{pmatrix} c_{p1} \\ c_{p2} \\ c_{p3} \\ . \\ . \\ . \\ c_{pp} \end{pmatrix}$

The block linear multi-step method is said to be of order p if $\overrightarrow{C_0} = \overrightarrow{C_1} = \overrightarrow{C_2} = ...\overrightarrow{C_p} = 0, \overrightarrow{C_{p+1}} \neq 0$ and the local truncation error is expressed as $T_n = \overrightarrow{C_{p+1}} h^{p+1} y^{p+1}(x_n)$.

STABILITY REGIONS OF THE BLOCK LINEAR MULTI-STEP METHODS: To determined the absolute stability regions of the block methods, they are reformulated as General Linear Methods of Burage & Butcher (1980) where they used a partition $(s + r) \times (s + r)$ matrix containing A_1, A_2, B_1 and B_2 expressed in the form.

$$\begin{bmatrix} Y \\ y^{i-1} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \begin{bmatrix} hf(Y) \\ y^i \end{bmatrix} \quad i = 1, 2, ..., N \qquad ...9.0$$

$Y = [Y_{i-1}^1 ... Y_{i-1}^s Y_{i-1}^1 ... Y_{i-1}^s]^T, y = [Y_{i-1}^1 ... Y_{i-1}^s y_{i-1} ... y_i]^T$

where

$$A_1 = \begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix} \quad B_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & u & e-u \end{bmatrix} \quad A_2 = \begin{bmatrix} A & B \\ 0 & 0 \\ v^T & \omega^T \end{bmatrix} \quad B_2 = \begin{bmatrix} I & u & e-u \\ 0 & 0 & I \\ 0 & \theta & 1-\theta \end{bmatrix}$$

Applying (9.0) to the test equation $y' = \lambda y, x \geq 0 \quad \lambda \in C$ leads to the recurrence equation

$y^{[i-1]} = M(z) y^{[i]} \ i = 1,2,...,N-1, z = \lambda h$, where the stability matrix

$$M(z) = B_2 + zA_2(I - zA_1)^{-1} B_1 \qquad ...10$$

and the stability polynomial of the method

$$\rho(\eta, z) = \det(\eta I - M(z)) \ . \qquad ...11$$

The absolute stability region of the method is defined as

$A = x \in C \ : \ \rho(\eta, z) = 1 \Rightarrow |\eta| \leq 1$

For the purpose of this paper, the matrices A_1, A_2, B_1, B_2 are replaced by A, B, V and U respectively. The coefficients of these matrices indicate the relationship between the various numerical quantities that arise in the computation.

ANALYSIS OF THE BLOCK LINEAR MULTI-STEP METHODS: In this section, the analysis of the convergence, order and stability properties of the block linear multi-step methods is carried out for some block linear multi-step methods.

CONVERGENCE ANALYSIS: Considering the block hybrid Adams Moulton method for $k = 4$ with one off grid point at $\alpha = \frac{7}{2}$ as given below:

$$\left.\begin{aligned}
y_{n+1} &= y_{n+3} + \tfrac{h}{90}[f_n - 34f_{n+1} - 114f_{n+2} - 34f_{n+3} + 1728f_{n+\frac{7}{2}} + f_{n+4}] \\
y_{n+2} &= y_{n+3} + \tfrac{h}{2520}[-11f_n + 105f_{n+1} - 1211f_{n+2} - 1981f_{n+3} + 704f_{n+\frac{7}{2}} - 126f_{n+4}] \\
y_{n+3} &= y_n + \tfrac{h}{280}[87f_n + 399f_{n+1} + 147f_{n+2} - 357f_{n+3} - 192f_{n+\frac{7}{2}} + 42f_{n+4}] \\
y_{n+\frac{7}{2}} &= y_{n+3} + \tfrac{h}{161280}[-83f_n + 6727f_{n+1} - 3038f_{n+2} + 44072f_{n+3} + 41600f_{n+\frac{7}{2}} - 2583f_{n+4}] \\
y_{n+4} &= y_{n+3} + \tfrac{h}{2520}[f_n - 7f_{n+1} + 21f_{n+2} + 371f_{n+3} + 1728f_{n+\frac{7}{2}} + 406f_{n+4}]
\end{aligned}\right\} \quad ...12.0$$

The block method (12) is represented in the form (2.0) to give

$$\text{h}=\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{-34}{90} & \frac{-114}{90} & \frac{-34}{90} & \frac{1728}{90} & \frac{1}{90} \\ \frac{105}{2520} & \frac{1211}{2520} & \frac{-1981}{2520} & \frac{704}{2520} & \frac{-126}{2520} \\ \frac{399}{280} & \frac{147}{280} & \frac{-357}{280} & \frac{-192}{280} & \frac{42}{280} \\ \frac{6725}{161280} & \frac{-3038}{161280} & \frac{44072}{161280} & \frac{44600}{161280} & \frac{-2583}{161280} \\ \frac{-7}{2520} & \frac{21}{2520} & \frac{371}{2520} & \frac{1728}{2520} & \frac{406}{2520} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & 0 & \frac{-11}{2520} \\ 0 & 0 & 0 & 0 & \frac{87}{280} \\ 0 & 0 & 0 & 0 & \frac{-83}{161280} \\ 0 & 0 & 0 & 0 & \frac{-7}{2520} \end{bmatrix} \begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

Where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & 0 & \frac{-11}{2520} \\ 0 & 0 & 0 & 0 & \frac{87}{280} \\ 0 & 0 & 0 & 0 & \frac{-83}{161280} \\ 0 & 0 & 0 & 0 & \frac{-7}{2520} \end{bmatrix}$$

The characteristic polynomial corresponding to (12.0) is given as:

$$\rho(R) = \det\{RA^0 - A^1\} = \det\left\{R\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\right\} = \det\begin{bmatrix}R&0&0&0&0\\0&R&0&0&0\\0&0&R&0&-1\\0&0&0&R&0\\0&0&0&0&R\end{bmatrix}$$

$$= R^5 \Rightarrow R = 0$$

By (4.0) the block method (12.0) is zero stable and consistent since its order $p = 6 \geq 1$ hence by Henrici (1962), the block method (12.0) is convergent.

ORDER OF THE BLOCK METHOD: Consider the block hybrid Adams Moulton Method for k=3 with one off grid point at $\mu = \frac{5}{2}$ as given below:

$$\left.\begin{aligned} y_{n+1} &= y_{n+2} + \tfrac{h}{1800}[31f_n - 755f_{n+1} - 1635f_{n+2} + 704f_{n+\frac{5}{2}} - 145f_{n+3}] \\ y_{n+2} &= y_n + \tfrac{h}{225}[71f_n + 320f_{n+1} + 15f_{n+2} + 64f_{n+\frac{5}{2}} - 20f_{n+3}] \\ y_{n+\frac{5}{2}} &= y_{n+2} + \tfrac{h}{28800}[37f_n - 335f_{n+1} + 7455f_{n+2} + 7808f_{n+\frac{5}{2}} - 565f_{n+3}] \\ y_{n+3} &= y_{n=2} + \tfrac{h}{1800}[-f_n + 5f_{n+1} + 285f_{n+2} + 121f_{n+\frac{5}{2}} + 295f_{n+3}] \end{aligned}\right\} \quad \ldots 13.0$$

Applying the method described for the order of the block LMM gives

$$\overrightarrow{C_0} = \overrightarrow{\alpha_0} + \overrightarrow{\alpha_1} + \overrightarrow{\alpha_2} + \overrightarrow{\alpha_{\frac{5}{2}}} + \overrightarrow{\alpha_3}$$

$$= \begin{bmatrix}0\\-1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\1\\-1\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\overrightarrow{C_1} = (\overrightarrow{\alpha_1} + 2\overrightarrow{\alpha_2} + \frac{5}{2}\overrightarrow{\alpha_{\frac{5}{2}}} + 3\overrightarrow{\alpha_3}) - (\overrightarrow{\beta_0} + \overrightarrow{\beta_1} + \overrightarrow{\beta_2} + \overrightarrow{\beta_{\frac{5}{2}}} + \overrightarrow{\beta_3})$$

$$= \left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\-2\\2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\\frac{5}{2}\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right\} - \left\{\begin{bmatrix}\frac{31}{1800}\\\frac{71}{225}\\\frac{37}{28800}\\\frac{-1}{1800}\end{bmatrix} + \begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\end{bmatrix} + \begin{bmatrix}\frac{-1635}{1800}\\\frac{15}{225}\\\frac{7455}{28800}\\\frac{285}{1800}\end{bmatrix} + \begin{bmatrix}\frac{704}{1800}\\\frac{64}{225}\\\frac{7808}{28800}\\\frac{121}{1800}\end{bmatrix} + \begin{bmatrix}\frac{-145}{1800}\\\frac{-20}{225}\\\frac{-565}{28800}\\\frac{295}{1800}\end{bmatrix}\right\} = \left(\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right)$$

$$\overrightarrow{C_2} = \frac{1}{2!}\left(\overrightarrow{\alpha_1} + 2^2\overrightarrow{\alpha_2} + \left(\frac{5}{2}\right)^2 \overrightarrow{\alpha_{\frac{5}{2}}} + 3^2\overrightarrow{\alpha_3}\right) - \left(\overrightarrow{\beta_1} + 2\overrightarrow{\beta_2} + \frac{5}{2}\overrightarrow{\beta_{\frac{5}{2}}} + 3\overrightarrow{\beta_3}\right)$$

$$= \left(\frac{1}{2!}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-4\\4\\-4\\-4\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{25}{4}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\9\\0\end{bmatrix}\right) - \left(\begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-3270}{1800}\\\frac{30}{225}\\\frac{14910}{28800}\\\frac{570}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{1760}{1800}\\\frac{160}{225}\\\frac{19520}{28800}\\\frac{3040}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-435}{1800}\\\frac{-60}{225}\\\frac{-1695}{28800}\\\frac{885}{1800}\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

$$\overrightarrow{C_3} = \frac{1}{3!}\left(\overrightarrow{\alpha_1} + 2^3\overrightarrow{\alpha_2} + \frac{5^3}{2}\overrightarrow{\alpha_{\frac{5}{2}}} + 3^3\overrightarrow{\alpha_3}\right) - \frac{1}{2!}\left(\overrightarrow{\beta_1} + 2^2\overrightarrow{\beta_2} + \left(\frac{5}{2}\right)^2\overrightarrow{\beta_{\frac{5}{2}}} + 3^2\overrightarrow{\beta_3}\right)$$

$$= \left(\frac{1}{3!}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-8\\8\\-8\\-8\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{125}{8}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\27\\0\end{bmatrix}\right) - \frac{1}{2!}\left(\begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-6540}{1800}\\\frac{60}{225}\\\frac{29820}{28800}\\\frac{1140}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{4400}{1800}\\\frac{400}{225}\\\frac{48800}{28800}\\\frac{7600}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-1305}{1800}\\\frac{-180}{225}\\\frac{-5085}{28800}\\\frac{2655}{1800}\\0\end{bmatrix}\right) = \left(\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right)$$

$$\overrightarrow{C_4} = \frac{1}{4!}\left(\overrightarrow{\alpha_1} + 2\overrightarrow{\alpha_2} + \left(\frac{5}{2}\right)^4 \overrightarrow{\alpha_{\frac{5}{2}}} + 3^4\overrightarrow{\alpha_3}\right) - \frac{1}{3!}\left(\overrightarrow{\beta_1} + 2^3\overrightarrow{\beta_2} + \left(\frac{5}{2}\right)^3\overrightarrow{\beta_{\frac{5}{2}}} + 3^3\overrightarrow{\beta_3}\right)$$

$$\left(\frac{1}{4!}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-16\\16\\-16\\-16\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{625}{16}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\81\\0\end{bmatrix}\right) - \frac{1}{3!}\left(\begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-13080}{1800}\\\frac{120}{225}\\\frac{59640}{28800}\\\frac{2280}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{11000}{1800}\\\frac{1000}{225}\\\frac{122000}{28800}\\\frac{19000}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-3915}{1800}\\\frac{-540}{225}\\\frac{-15255}{28800}\\\frac{79565}{1800}\\0\end{bmatrix}\right) = \left(\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right)$$

$$\overrightarrow{C_5} = \frac{1}{5!}\left(\overrightarrow{\alpha_1} + 2^5\overrightarrow{\alpha_2} + \left(\frac{5}{2}\right)^5 \overrightarrow{\alpha_{\frac{5}{2}}} + 3^5\overrightarrow{\alpha_3}\right) - \frac{1}{4!}\left(\overrightarrow{\beta_1} + 2^4\overrightarrow{\beta_2} + \left(\frac{5}{2}\right)^4\overrightarrow{\beta_{\frac{5}{2}}} + 3^4\overrightarrow{\beta_3}\right)$$

$$\left(\frac{1}{5!}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-32\\32\\-32\\-32\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{3125}{32}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\243\\0\end{bmatrix}\right) - \frac{1}{4!}\left(\begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-26160}{1800}\\\frac{240}{225}\\\frac{119280}{28800}\\\frac{4560}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{27500}{1800}\\\frac{2500}{225}\\\frac{305000}{28800}\\\frac{47500}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-11745}{1800}\\\frac{-1620}{225}\\\frac{-45765}{28800}\\\frac{23895}{1800}\\0\end{bmatrix}\right) = \left(\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right)$$

$$\overrightarrow{C_6} = \frac{1}{6!}\left(\overrightarrow{\alpha_1} + 2^6\overrightarrow{\alpha_2} + \left(\frac{5}{2}\right)^6 \overrightarrow{\alpha_{\frac{5}{2}}} + 3^6\overrightarrow{\alpha_3}\right) - \frac{1}{5!}\left(\overrightarrow{\beta_1} + 2^5\overrightarrow{\beta_2} + \left(\frac{5}{2}\right)^5\overrightarrow{\beta_{\frac{5}{2}}} + 3^5\overrightarrow{\beta_3}\right)$$

$$\left(\frac{1}{6!}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-64\\64\\-64\\-64\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\frac{15625}{64}\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\729\\0\end{bmatrix}\right) - \frac{1}{5!}\left(\begin{bmatrix}\frac{-755}{1800}\\\frac{320}{225}\\\frac{-335}{28800}\\\frac{5}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-52320}{1800}\\\frac{480}{225}\\\frac{238560}{28800}\\\frac{9120}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{68750}{1800}\\\frac{6250}{225}\\\frac{762500}{28800}\\\frac{118750}{1800}\\0\end{bmatrix} + \begin{bmatrix}\frac{-35235}{1800}\\\frac{-4860}{225}\\\frac{-137295}{28800}\\\frac{71685}{1800}\\0\end{bmatrix}\right) = \begin{bmatrix}\frac{11}{360}\\\frac{7}{900}\\\frac{83}{230400}\\\frac{-1}{3600}\\0\end{bmatrix}$$

Since $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0$ and $C_6 \neq 0$, the block hybrid method (13.0) for $k = 3$ with one off grid point is of order $p = 5$ and its error constant is

$$T_n = \begin{bmatrix} \frac{11}{360} \\ \frac{7}{900} \\ \frac{83}{230400} \\ \frac{-1}{3600} \\ 0 \end{bmatrix}$$

Absolute stability regions of the block methods: To plot the absolute stability regions of the block hybrid Adams Moulton method for k=3, equation (13.0) is expressed in the form (9.0) to yield the table below:

$$\left\{\begin{array}{c} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ --- \\ y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{array}\right\} = \left\{\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{31}{1800} & \frac{-755}{1800} & \frac{-1635}{1800} & \frac{704}{1800} & \frac{-145}{1800} & 1 & 0 & 0 \\ \frac{71}{225} & \frac{320}{225} & \frac{15}{225} & \frac{64}{225} & \frac{-20}{225} & 0 & 0 & 1 \\ \frac{37}{28800} & \frac{-335}{28800} & \frac{7455}{28800} & \frac{7808}{28800} & \frac{-565}{28800} & 1 & 0 & 0 \\ \frac{-1}{1800} & \frac{5}{1800} & \frac{285}{1800} & \frac{121}{1800} & \frac{295}{1800} & 1 & 0 & 0 \\ --- & --- & --- & --- & --- & --- & --- & --- \\ \frac{-1}{1800} & \frac{5}{1800} & \frac{285}{1800} & \frac{121}{1800} & \frac{295}{1800} & 1 & 0 & 0 \\ \frac{71}{225} & \frac{71}{225} & \frac{71}{225} & \frac{71}{225} & \frac{71}{225} & 0 & 0 & 1 \\ \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} & 1 & 0 & 0 \end{array}\right\} \left\{\begin{array}{c} hf_n \\ hf_{n+1} \\ hf_{n+2} \\ hf_{n+\frac{5}{2}} \\ hf_{n+3} \\ --- \\ y_{n+2} \\ y_{n+1} \\ y_n \end{array}\right\}$$

Where

$$A = \left\{\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \frac{31}{1800} & \frac{-755}{1800} & \frac{-1635}{1800} & \frac{704}{1800} & \frac{-145}{1800} \\ \frac{71}{225} & \frac{320}{225} & \frac{15}{225} & \frac{64}{225} & \frac{-20}{225} \\ \frac{37}{28800} & \frac{-335}{28800} & \frac{7455}{28800} & \frac{7808}{28800} & \frac{-565}{28800} \\ \frac{-1}{1800} & \frac{5}{1800} & \frac{285}{1800} & \frac{121}{1800} & \frac{295}{1800} \end{array}\right\} \quad U = \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right\}, \quad B = \left\{\begin{array}{ccccc} \frac{-1}{1800} & \frac{5}{1800} & \frac{285}{1800} & \frac{121}{1800} & \frac{295}{1800} \\ \frac{71}{225} & \frac{71}{225} & \frac{71}{225} & \frac{71}{225} & \frac{71}{225} \\ \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} & \frac{31}{1800} \end{array}\right\}, V = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right\}$$

Substituting the values of the matrices A, B, U and V into the stability matrix (10.0) and the stability function (11.0) and using the maple software produces the stability polynomial of the method. The stability polynomial is used in a matlab environment and produces the region of absolute stability of the block method k=3 as shown in Fig. 1

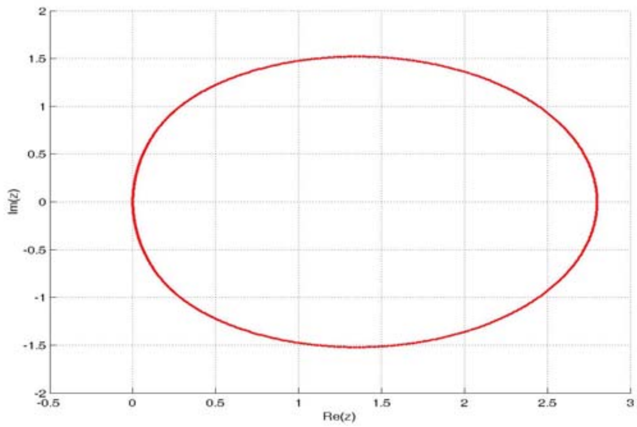

FIG1. THE ABSOLUTE STABILITY REGION OF THE BHAM K=3 WITH ONE OFF GRID POINT

CONCLUSION

The convergence, order and region of absolute stability of block linear multi-step methods including its hybrid forms have been determined for the case $k=3$ and $k=4$. These examples demostrate that the new approaches have resolved the difficulties of determining the convergence, order and absolute stability regions of block linear multi-step methods. The stability region displayed in figure 1 showed that the Block linear multi-step method for k=3 is A -Stable and is of order $p=5$. while the method $k=4$ is convergent. Future work will include implementation issues such as error estimates and step size control and changing strategy.

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