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PARAMETRIC REACCELERATED OVERRELAXATION (PROR) METHOD FOR NUMERICAL SOLUTION OF LINEAR SYSTEMS

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ABSTRACT
This paper proposes the Parameterized Reaccelerated Overrelaxation (PROR) method for numerical solution of linear systems arising from the discretization of partial differential equations. The method is a three-parameter generalization of the Reaccelerated overrelaxation (ROR) method. An expression for the eigenvalues of the iteration matrix of the method is obtained in terms of the eigenvalues of the corresponding Jacobi iteration matrix. Functional relations for determining the optimum values of the parameters are established. Numerical examples are presented to validate theoretical results as well as compare with existing methods. Results showed that the method is suitable and compares favourably with AOR, ROR and PAOR methods.

Keywords: AOR, ROR, PAOR, spectral radius, Convergence.

INTRODUCTION
Partial differential equations (PDEs) play a significant role in many problems in Mathematics, Physics, Engineering, Economics and other Science and non-Science related fields. The Poisson equation, for example, occurs frequently in electromagnetism, fluid dynamics among others. Often, it is easy to express PDEs as systems of linear equations of the form

$$Bx = c \tag{1}$$

The usual splitting of B in (1) gives

$$(D - L_B - U_B)x = c \tag{2}$$

where $D, -L_B, -U_B$ are the diagonal, strictly lower and strictly upper parts of B respectively.

Although, it may be possible to use direct methods such as matrix inversion, Gaussian elimination, and so on, to exactly solve (1), the direct methods tend to take far too long to be practicable for very large and/or sparse systems. This difficulty in using direct methods for solving such systems of linear equations makes a case for iterative methods.
In the past few decades, development of iterative methods for the solution of partial differential equations resulting in (1) have evolved, methods such as Gauss-Seidel (1874) and Jacobi (1884) have been introduced. Young (1950) introduced the SOR method which is an extrapolated Gauss-Seidel method that gives faster convergence than the Jacobi and Gauss-Seidel methods. Hadjidimos (1978) introduced the AOR, a two-parameter generalization of the SOR method which also gives better convergence results than the SOR. Thereafter, several modifications of the SOR and AOR methods have been made in an attempt to speed up the rate of convergence of the methods. These include Avdelas and Hadjimos (1981) who optimized the AOR method for the special case when matrix B in (1) is consistently ordered. Wu and Liu (2014) proposed a new version of the AOR named the quasi accelerated overrelaxation method (QAOR), Youssef and Farid (2015) derived another variant of the AOR called the KAOR. Most recently, Vatti *et al* (2020) proposed two different versions of the AOR called parametric accelerated overrelaxation (PAOR) method and the reaccelerated overrelaxation (ROR) method derived for consistently ordered matrices. This present work is aimed at further improving the convergence of the AOR method by proposing a new version of the method named PROR

MATERIALS AND METHODS
Derivation of the Method
Consider the linear system (2) given by:

$$(D - L_B - U_B)x = c \tag{3}$$

which results in

$$(I - L - U)x = b \tag{4}$$

or equivalently

$$Ax = b \tag{5}$$

where $A = I - L - U,\ \ b = D^{-1}c,\ \ L = D^{-1}L_B,\ \ U = D^{-1}U_B$.

The accelerated over-relaxation method (AOR) for solving (5) is given by

$$x^{(n+1)} = (I - \omega L)^{-1}\{[(1-r)I + (r-\omega)L + rU]x^{(n)} + rb\,\} \tag{6}$$

while the Parametric accelerated over-relaxation (PAOR) method is as follows:

$$x^{(n+1)} = [(1+\alpha)I - \omega L)]^{-1}\{[(1+\alpha-r)I + (r-\omega)L + (r)U]x^{(n)} + rb\,\} \tag{7}$$

where α is a fixed parameter, $\alpha \neq -1$

and the reaccelerated overrelaxation (ROR) method is given as:

$$x^{(n+1)} = (I - \omega L)^{-1}\{[(1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U]x^{(n)} + (r - r\omega)b\} \quad (8)$$

Considering the parameters $r \neq 0$ and $\omega \neq 0$ and adding $r\omega(Ax - b) = 0$ to (7), gives

$$x^{(n+1)} = [(1+\alpha)I - \omega L)]^{-1}\{[(1 + \alpha - r)I + (r - \omega)L + (r)U]x^{(n)} + rb + r\omega[(I - L - U)x^{(n)} - b]\} \quad (9)$$

$$\Rightarrow x^{(n+1)} = [(1+\alpha)I - \omega L)]^{-1}\{[(1 + \alpha - r)I + (r - \omega)L + (r)U]x^{(n)} + rb + (r\omega I - r\omega L - r\omega U)x^{(n)} - r\omega b]\} \quad (10)$$

Simplifying (10), we obtain our proposed Parametric reaccelerated over-relaxation (PROR) scheme denoted by $M_{\alpha r,\omega}$, as:

$$x^{(n+1)} = [(1+\alpha)I - \omega L)]^{-1}\{[(1 + \alpha - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U]x^{(n)} + (r - r\omega)b\} \quad (11)$$

where the iterative matrix of the PROR scheme, $L_{\alpha,r,\omega}$ is given as

$$L_{\alpha,r,\omega} = [(1+\alpha)I - \omega L)]^{-1}\{[(1 + \alpha - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U] \quad (12)$$

Method (11) reduces to some known methods for specific choices of the parameters (α, r, ω) as follows:
$M_{0,r,\omega}$ gives the Reaccelerated over-relaxation (ROR) method, i. e. for $(\alpha, r, \omega) = (0, r, \omega)$)
$M_{0,1,0}$ gives the Jacobi method i. e. for $(\alpha, r, \omega) = (0,1,0)$

Convergence Analysis of PROR Method

Consistently Ordered Matrix: Let A be a consistently ordered square matrix. That is, A is a matrix for which the expression $|aA_L + a^{-1}A_U - bD|$ is independent of a for $a \neq 0$ and for all b. where A_L, A_U and D are the strictly lower, upper and diagonal matrices of A respectively. Then, the following theorems are proposed and proved.

Theorem 1: If λ is the eigenvalue of the iteration matrix $L_{\alpha,r,\omega}$ of the PROR method. Then, the characteristic equation of $L_{\alpha,r,\omega}$ is represented as :

$$|[(1+\alpha)I - \omega L)]\lambda I - \{[(1 + \alpha - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U]\,| = 0 \quad (13)$$

Proof: Since λ is an eigenvalue of the iteration matrix $L_{\alpha,r,\omega}$, then we have the characteristic equation given as

$$|\lambda I - L_{\alpha,r,\omega}| = 0 \quad (14)$$

Substituting $L_{\alpha,r,\omega}$ given in (12) into (14), we obtain

$$|\lambda I - [(1+\alpha)I - \omega L)]^{-1}\{[(1 + \alpha - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U]| = 0 \quad (15)$$

$$\Rightarrow |[(1+\alpha)I - \omega L)]\lambda I - \{[(1 + \alpha - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U]| = 0 \quad (16)$$

which completes the required proof

Theorem 2: Let λ be the eigenvalue of the iteration matrix $L_{\alpha,r,\omega}$ of the PROR method and μ be the eigenvalue of the corresponding Jacobi iteration matrix $L_{0,1,0}$. Then, λ and μ are connected by the relation:

$$[(1+\alpha)\lambda + \mathrm{r}(1-\omega) - (1+\alpha))]^2 = r\omega\mu^2(1-\omega)\lambda + r^2\mu^2(1-\omega)^2 - r\mu^2\omega(1-\omega) \quad (17)$$

Proof: Form theorem 1, we have that

$$|[(1+\alpha)\lambda I - \omega\lambda L - (1 + \alpha - r + r\omega)I - [(r - \omega - rw)L - (r - r\omega)U]| = 0 \quad (18)$$

$$\Rightarrow |[(1+\alpha)\lambda-(1+\alpha-r+r\omega)]I-[(\omega\lambda+r-\omega-rw)L+(r-r\omega)U]|=0 \quad (19)$$

$$\left|\left[\frac{(1+\alpha)\lambda-(1+\alpha-r+r\omega)}{(\omega\lambda+r-\omega-rw)^{\frac{1}{2}}(r-r\omega)^{\frac{1}{2}}}\right]I-(L+U)\right|=0 \quad (20)$$

We note that $L+U$ is the Jacobi iteration matrix, $L_{0,1,0}$. Thus, (20) is the characteristic equation of $L_{0,1,0}$ and since μ is the eigenvalue of $L_{0,1,0}$, we have that

$$\frac{(1+\alpha)\lambda-(1+\alpha-r+r\omega)}{(\omega\lambda+r-\omega-rw)^{\frac{1}{2}}(r-r\omega)^{\frac{1}{2}}}=\mu \quad (21)$$

That is

$$[(1+\alpha)\lambda-(1+\alpha-r+r\omega)]^2=\mu^2(\omega\lambda+r-\omega-rw).(r-r\omega) \quad (22)$$

and so

$$[(1+\alpha)\lambda+\mathrm{r}(1-\omega)-(1+\alpha))]^2=r\omega\mu^2(1-\omega)\lambda+r^2\mu^2(1-\omega)^2-r\mu^2\omega(1-\omega) \quad (23)$$

Theorem 3: Suppose λ and μ are the eigenvalues of $L_{\alpha,r,\omega}$ and $L_{0,1,0}$ respectively, then

$$\lambda=\frac{r\mu^2\omega(1-\omega)}{2(1+\alpha)^2}-\frac{r(1-\omega)}{1+\alpha}+1\,,\qquad \text{for } \omega=\frac{2(1+\alpha)}{1+\sqrt{1-\mu^2}} \quad (24)$$

Proof: From (24), we have

$$[(1+\alpha)\lambda+\mathrm{r}(1-\omega)-(1+\alpha))]^2=r\omega\mu^2(1-\omega)\lambda+r^2\mu^2(1-\omega)^2-r\mu^2\omega(1-\omega) \quad (25)$$

Expanding (25), we obtain

$$[(1+\alpha)\lambda+(\mathrm{r}-\mathrm{r}\omega-(1+\alpha))]^2=r\omega\mu^2(1-\omega)\lambda+r^2\mu^2(1-\omega)^2-r\mu^2\omega(1-\omega) \quad (26)$$

$$=(1+\alpha)^2\lambda^2-[\omega(\mathrm{r}-\mathrm{r}\omega)\mu^2-2(1+\alpha)(\mathrm{r}-\mathrm{r}\omega-(1+\alpha))]\lambda+[\mathrm{r}-\mathrm{r}\omega-(1+\alpha)]^2+\omega(r-r\omega)\mu^2-(r-r\omega)^2\mu^2 = 0 \quad (27)$$

It is observed that (27) is a quadratic equation in λ. Thus, the solution is obtained for λ as:

$$\lambda=\frac{r\mu^2\omega(1-\omega)-2(1+\alpha)[\mathrm{r}(1-\omega)-(1+\alpha)]}{2(1+\alpha)^2}\pm\frac{\sqrt{\Delta}}{2(1+\alpha)^2} \quad (28)$$

where

$$\Delta=\omega^2(\mathrm{r}-\mathrm{r}\omega)^2\mu^4-4(1+\alpha)[\mathrm{r}(1-\omega)-(1+\alpha)]\omega(\mathrm{r}-\mathrm{r}\omega)\mu^2+4(1+\alpha)^2[\mathrm{r}-\mathrm{r}\omega-(1+\alpha)]^2-4(1+\alpha)^2[\mathrm{r}-\mathrm{r}\omega-(1+\alpha)]^2-4(1+\alpha)^2\omega(\mathrm{r}-\mathrm{r}\omega)\mu^2+4(1+\alpha)^2(\mathrm{r}-\mathrm{r}\omega)^2\mu^2 \quad (29)$$

$$=(\mathrm{r}-\mathrm{r}\omega)^2\{\omega^2\mu^2-4(1+\alpha)\omega+4(1+\alpha)^2\} \quad (30)$$

We note that (30) equals zero if

$$\omega^2\mu^2-4(1+\alpha)\omega+4(1+\alpha)^2=0 \quad (31)$$

That is,

$$\omega=\frac{2(1+\alpha)}{1+\sqrt{1-\mu^2}}\quad or \qquad \omega=\frac{2(1+\alpha)}{1-\sqrt{1-\mu^2}} \quad (32)$$

Thus if (32) holds, then (28) becomes

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$$\lambda = \frac{r\mu^2\omega(1-\omega) - 2(1+\alpha)[\mathrm{r}(1-\omega)-(1+\alpha)]}{2(1+\alpha)^2} \quad (33)$$

i.e. $\lambda = \frac{r\mu^2\omega(1-\omega)}{2(1+\alpha)^2} - \frac{r(1-\omega)}{1+\alpha} + 1$ (34)

Choice of Parameters, α, r and ω

We recall the eigenvalues of the iteration matrix of the PROR method given in terms of those of the corresponding Jacobi iteration matrix in (34) as below:

$$\lambda = \frac{r\mu^2\omega(1-\omega)}{2(1+\alpha)^2} - \left(\frac{r(1-\omega)}{1+\alpha} - 1\right) \quad (35)$$

Let (35) be such that

$$\frac{r\mu^2\omega(1-\omega)}{2(1+\alpha)^2} = k\left(\frac{r(1-\omega)}{1+\alpha} - 1\right) \quad (36)$$

where k is any real constant, $k \neq 0$

Now, expressing k in terms of λ and vice-versa, we obtain the following relations:

$$\frac{r\mu^2\omega(1-\omega)}{2(1+\alpha)^2} = \frac{kr(1-\omega)}{1+\alpha} - k \quad (37)$$

$$\Rightarrow r = \frac{(1+\alpha)k}{(1-\omega)[k - \frac{\omega\mu^2}{2(1+\alpha)}} \quad (38)$$

and

$$k = \frac{r\mu^2\omega(1-\omega)}{1+\alpha} \cdot \frac{1}{2[r(1-\omega)-(1+\alpha)]} \quad (39)$$

Now, from (38) and (39) we can obtain expressions for $k(1+\alpha)$ as follows:

From (38)

$$k(1+\alpha) = r\left[k(1-\omega) - \frac{\omega\mu^2(1-\omega)}{2(1+\alpha)}\right] \quad (40)$$

and from (39), we have

$$k(1+\alpha) = \frac{r\omega\mu^2(1-\omega)}{2[r(1-\omega)-(1+\alpha)]} \quad (41)$$

Equating (40) and (41) gives

$$[r(1-\omega)-(1+\alpha)]\left[k(1-\omega) - \frac{\omega\mu^2(1-\omega)}{2(1+\alpha)}\right] = \frac{\omega\mu^2(1-\omega)}{2} \quad (42)$$

Consider the expression

$$\frac{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}}{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}} \tag{43}$$

where $\bar{\mu}^2$ and $\underline{\mu}^2$ are the maximum and minimum among the absolute values of μ respectively. In view of (43), we can obtain the following equation from (42):

$$[r(1-\omega)-(1+\alpha)]\left[k(1-\omega)-\frac{\omega\mu^2(1-\omega)}{2(1+\alpha)}\right] = \left[\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}\right]\left[\frac{\frac{\omega\mu^2(1-\omega)}{2}}{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}}\right] \tag{44}$$

so that

$$r(1-\omega)-(1+\alpha) = \omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2} \tag{45}$$

$$\Rightarrow r = \left(1+\alpha+\omega+\frac{\bar{\mu}^2 - \underline{\mu}^2}{2}\right)\frac{1}{(1-\omega)} \tag{46}$$

and

$$k(1-\omega)-\frac{\omega\mu^2(1-\omega)}{2(1+\alpha)} = \frac{\frac{\omega\mu^2(1-\omega)}{2}}{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}} \tag{47}$$

$$\Rightarrow k = \frac{\omega\mu^2}{2(1+\alpha)} + \frac{\frac{\omega\mu^2}{2}}{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}} \tag{48}$$

Substituting $\omega = \frac{2(1+\alpha)}{1+\sqrt{1-\bar{\mu}^2}}$ into (47), we have an expression for k as

$$k = 1-\sqrt{1-\bar{\mu}^2} + \frac{\frac{\omega\mu^2}{2}}{\omega + \frac{\bar{\mu}^2 - \underline{\mu}^2}{2}} \tag{49}$$

Following Vatti *et al* (2020), if $k > 1$ in (49), then r is as given in (46) but if $k < 1$ in (49), then r is taken to be half of the value given in (46).

In summary, we categorize the choice for optimum parameters α, r and ω presented above into three cases:

Case I: When $\underline{\mu} = \bar{\mu}$ and $k = 1$

then $\omega = \frac{2(1+\alpha)}{1+\sqrt{1-\bar{\mu}^2}}$, $r = \left(\frac{1+\alpha}{\sqrt{1-\bar{\mu}^2}}\right)\left(\frac{1}{1-\omega}\right)$

Case II: When $\underline{\mu} \neq \bar{\mu}$ and $k > 1$

then $\omega = \frac{2(1+\alpha)}{1+\sqrt{1-\bar{\mu}^2}}$, $r = \left(1+\alpha+\omega+\frac{\bar{\mu}^2-\underline{\mu}^2}{2}\right)\left(\frac{1}{1-\omega}\right)$

Case III: When $\underline{\mu} \neq \bar{\mu}$ and $k < 1$

then $\omega = \frac{2(1+\alpha)}{1+\sqrt{1-\bar{\mu}^2}}$, $r = \left(1+\alpha+\omega+\frac{\bar{\mu}^2-\underline{\mu}^2}{2}\right)\frac{\left(\frac{1}{1-\omega}\right)}{2}$

It is noteworthy that $\underline{\mu}$ can be equal to zero in all the three cases.

RESULTS AND DISCUSSION

Problem 1: Consider the consistently ordered matrix $\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ found in Hadjidimos (1978)

$$\underline{\mu} = \bar{\mu} = \frac{2\sqrt{2}}{3}$$

Problem 2: For the matrix $A = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{71}{10} & \frac{113}{10} \\ \frac{16}{5} & \frac{1}{5} & 1 & 0 \\ 2 & \frac{1}{5} & 0 & 1 \end{bmatrix}$

and $b = \begin{bmatrix} \frac{7}{5} \\ \frac{26}{5} \\ \frac{22}{5} \\ \frac{16}{5} \end{bmatrix}$

$$\underline{\mu} = \frac{\sqrt{23}}{5}, \bar{\mu} = \frac{\sqrt{24}}{5}$$

considered by Vatti et al (2020) and Avdelas and Hadjimos (1981) The results of the above examples obtained from the optimum values of parameters for AOR, ROR, PAOR and PROR methods are presented as comparative analysis in tables 1 and 2.

Table 1: Convergence Results for Problem 1

Method	Choice of Parameters	Number of iterations	Spectral radius
AOR	$r = 3, \omega = \frac{3}{2}$	2	0
ROR	$r = -6, \omega = \frac{3}{2}$	2	0
PAOR	$\alpha = 1, r = 6, \omega = 3$	2	0
PROR	$\alpha = 1, r = -3, \omega = 3$	2	0

Table 2: Convergence Results for Problem 2

Method	Choice of Parameters	Number of iterations	Spectral radius
AOR	$r = \frac{-5}{4}, \omega = \frac{5}{3}$	diverging	1.3070322618
AOR	$r = \frac{14}{3}, \omega = \frac{5}{3}$	86	0.7512951780
ROR	$r = \frac{-403}{100}, \omega = \frac{5}{3}$	45	0.5689256932
PAOR	$\alpha = \frac{-9}{10}, r = \frac{43}{150}, \omega = \frac{1}{6}$	44	0.5653710679
PROR	$\alpha = \frac{-9}{10}, r = \frac{43}{125}, \omega = \frac{1}{6}$	44	0.5653710679

Table 1 and 2 display the spectral radii as well as the number of iterations needed to reach an accuracy of about ten decimal places (tolerance) for the case of optimum values of the parameters computed using the functional relations for each of the methods for problems 1 and 2 respectively. The results revealed that the PROR method agrees with other methods in Table 1 with spectral radius equal to zero. In Table 2, it is observed that the spectral radius of the derived PROR coincides with that of PAOR while it maintains a lead over AOR and ROR methods, indicating faster convergence than the AOR and ROR methods. Worthy of note is that for the pair of optimum values of AOR parameters provided by Avdelas and Hadjimos (1981) for this problem, the spectral radius is 1.3070322618 and not 0.5651941652 as claimed in the paper which shows the method diverges for that pair. However, Vatti *et al* (2020) provided a different pair of parameter values for which the spectral radius of the AOR iteration matrix is 0.7512951780 which still lags behind the derived method PROR and the other versions of AOR methods considered, in terms of convergence.

Conclusion

This work has derived a new version of the AOR iterative method named PROR method which is a hybrid of the PAOR and ROR methods and determined the optimum choice of the parameters in order to speed up rate of convergence. It can be observed from the numerical examples considered above that the method competes admirably with recent versions of the AOR methods.

REFERENCES

Avdelas G. & Hadjidimos, A. (1981). Optimum accelerated overrelaxation method in a special case. *Mathematics of computation,* 36(153),183-187.

Hadjidimos, A. (1978). Accelerated overrelaxation method. *Mathematics of Computation*, (32) 141:149-157.

Jacobi, C. G. J. (1884). Gessamelte werke. Berlin: G. Reimer. pp 467.

Seidel, L. (1874). Abh bayer akad wiss. Naturwiss. Kl. 11(3), 81.

Vatti, V. B. K., Rao, G. C. & Pai S. S. (2020). Parametric overrelaxation (PAOR) method. *Numerical optimization in Engineering and Sciences, Advances in Intelligent Systems and Computing,* 979, 283-288.

Vatti, V. B. K., Rao, G. C. & Pai, S. S. (2020). Reaccelerated overrelaxation (ROR) method. *Bulettin of the International Mathematical Virtual Institute*, 10(2), 315-324.

Wu, S., & Liu, Y. (2014). A new version of accelerated overrelaxation iterative method. *Journal of Applied Mathematics, (725360)*, 1-6.

Young, D. M. (1950). Iterative methods for solving partial difference equations of elliptic type. Doctoral Thesis, Harvard University, Cambridge, MA. pp 74.

Youssef, I. K. & Farid, M. M. (2015). On the accelerated overrelaxation method. *Pure and Applied Mathematics Journal*, 4(1), 26 – 31.

Youssef, I. K. & Taha, A. A. (2013). On the modified successive overrelaxation method. Applied Mathematics & Computation., 219(9), 4601- 4613.

Youssef, I. K. (2012). On the successive overrelaxation method. *Journal of Mathematics and Statistics*, 8(2); 176-184..