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TOPOLOGICAL SPACES WITH AN EMPHASIS ON THE SPHERE AND ITS APPLICATIONS

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ABSTRACT
Algebraic topology has grown over the last decades. This paper presents the essentials of topological spaces, continuous functions and homotopy. The human heart is assumed to be topologically equivalent to the one-sphere. When the human heart undergoes a stimulus, the time in the beat cycle is mapped to the time it recovers from the stimulus. This work concluded with the computation of the stereographic projection and its applications.

Keywords: Continuous functions, Homotopy, Stereographic projection, Topological spaces.

1. INTRODUCTION

1.1

Topology is a mathematical discipline that studies the properties of shapes. The objects we study in topology are called topological spaces. The set W and the topology γ on it, denoted by (W, γ), is known as a ***topological space*** (Munkres, 2000). Considering the circle, written as S^1, in the complex plane of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. For the set of points of $\mathbb{R}$ in the euclidean plane;

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

However, the S^0 is two points, S^1 is the circle, S^2 is the sphere and S^3 is the torus. Then the number of n−sphere denoted

$$S^n = \left\{(x, y) \in \mathbb{R}^{n+1} : {x^2}_n + \cdots + {y^2}_{n+1} = 1\right\}$$

(Khatchatourian, 2018).

Homotopy is fundamental in algebraic topology. It is the continuous deformation of one function to another. The circle is a topological space which can be used to model the stimulus of a heartbeat. The time the stimulus occurs and beats again is known as the *Latency* denoted by L. The circle is topologically equivalent to the annulus. From the annulus we can have discontinuities which causes fibrillation. When the circle is deformed to itself, the number of times it winds around itself is called degree. Suppose two circle functions are homotopic, then their degrees are the same (Adams and Franzosa, 2008).

Again, the stereographic projection is a mapping that projects a two-sphere onto a plane or vice-versa.

1.2 Preliminaries

1.2.1

Let the set $X = \{a, b, c, d, e\}$ *and* $\Gamma = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$. *The set* U *is called* ***Open sets*** *if and only if* $U \subseteq X$ *and* $U \in \Gamma$.

1.2.2 Topological space

Given the set W and a topology γ *being a family of subsets known as open sets, such that:*

1. *The empty set* $\emptyset$ *and W are elements of* γ .
2. *The arbitrary union of the elements of any open sets of* γ *is in* γ.
3. *The finite intersection of elements of* γ *is in* γ.

The pair of set (W, γ), where γ is a collection of subsets of W, is known as **Topological space**. Without loss of generality, A set W is always a topological space; The **Trivial Topology**: $T_T = \{\emptyset, W\}$,
The **Discrete Topology**: $T_D = \{all\ subsets\ of\ W\}$ (Munkres, 2000), (Adams and Franzosa, 2008).

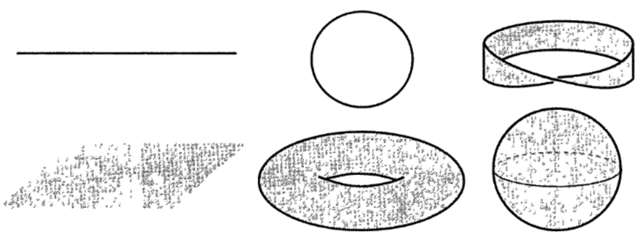

Figure 1: Examples of topological spaces

1.2.3 Continuous functions

Suppose (X, Γ) *and* (Y, U) *are topological spaces. Define the continuous map*
$f : X \rightarrow Y$ *such that the inverse image of all open subsets of* Y *is also open in* X*, that is,*
$f^{-1}(V) \in \Gamma$ *for every* $V \in U$*, then f is called a* ***Continuous function*** (Khatchatourian, 2018).

Example

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$ be two topological spaces with topologies Γ_1 and Γ_2 respectively. Where $\Gamma_1 = \{\emptyset, X, \{a, b\}, \{b\}, \{b, c\}\}$ and $\Gamma_2 = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}\}$
Given the map $f : X \rightarrow Y$ such that $f(a) = 1, f(b) = 1, f(c) = 3.$
$Now,\ the\ open\ sets\ in\ Y\ are\ \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}\}.$
$f^{-1}(\emptyset) = \emptyset,\ f^{-1}(Y) = X,\ f^{-1}(\{1\}) = \{a, b\},$
$f^{-1}(\{2\}) = \emptyset,\ f^{-1}(\{1, 2\}) = \{a, b\}.$
Clearly, as seen in Figure 2, the inverse image of every open set in Y is open in X. Therefore, its continuous.

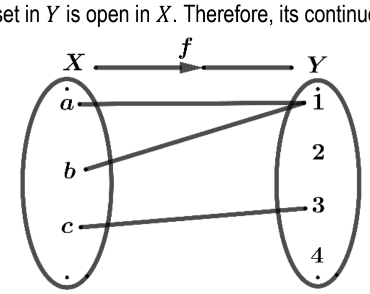

Figure 2: Continuous functions

1.2.4 Homotopy

If we consider two topological spaces X and Y, and define two continuous maps $f, g : X \rightarrow Y$ with the interval $I = [0, 1]$. The two maps are said to be homotopic such that there exists a continuous function

$$F : X \times I \rightarrow Y$$

*if and only if $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Then we can say that **f is homotopic to g**. In this case we write $F : f \simeq g$* (Adams and Franzosa, 2000).

1.2.5 Equivalence relation

Suppose X and Y are topological spaces, then the relation $f : X \rightarrow Y$ is an equivalence relation.

1.2.6 Circle function

A continuous function $f : S^1 \rightarrow S^1$ is known as a Circle function.

Theorem 1

Given any circle function $f : S^1 \rightarrow S^1$, there exist $n \in \mathbb{Z}$ where f is homotopic to $C_n(\theta) = \theta$ (Adams and Franzosa, 2008).

Corollary

The circle function in Theorem 7 has a unique n and is defined to be the degree of f which is written as $deg(f)$ (Adams and Franzosa, 2008).

Proposition

Let two given circle functions $f, g : S^1 \rightarrow S^1$, such that $deg(f) = deg(g)$ then the the two circle functions are homotopic.

Theorem 2

Suppose a function $f : S^1 \rightarrow S^1$ is a continuous and has nonzero degree, then we say that f is surjective (Adams and Franzosa, 2000).

1.2.7 Stereographic Projection

Definition 1

Suppose there is a unit sphere which sits in the complex plane such that there exists a point on the plane which relates to a unique point on the unit sphere, then we have a Stereographic Projection (Welke, 1997).

2.1 The Homotopy of the circle function

Homotopy is the continuous deformation of functions. Let the human heart be topologically equivalent to the circle. Let the time in the beat cycle be a variable θ on a circle and assume that the heart beats $\theta = 0$. Consider the heart beat interval of 2π. The stimulus response of the heart measured during the time the heart will beat again after stimulus is called *Latency* denoted by L. During the beat cycle, the time we apply the stimulus is known as coupling interval and denoted by c. Let us take stimuli of different intensity from weak ($s = w$) to strong ($s = g$) and analyze $L(c, s)$.

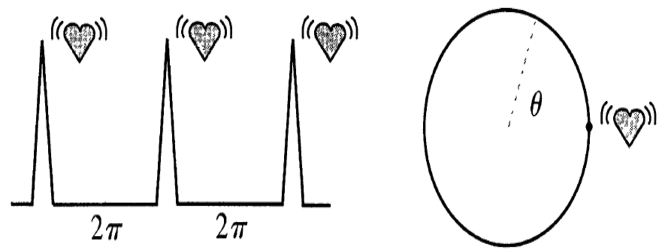

Figure 3: Time in the beat cycle

Taking the latency as a continuous function of coupling interval and stimulus strength, we have the function $L : S^1 \times [w, g] \rightarrow S^1$. We can draw two arguments:

1. $L(c, w) = 2\pi - c$ such that $c \in S^1$.
2. Suppose $\theta^* \in S^1$, there exist $L(c, g) = \theta^*$ such that $c \in S^1$.

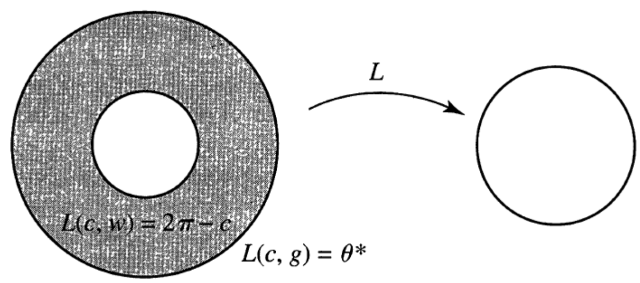

Figure 4: Latency defined over $S^1 \times [w, g]$

Suppose we restrict the Latency L to a specific stimulus strength s^* then the outcome is a circle function $L|_{S^1 \times \{s^*\}}$. Whence, $L : s^1 \times [w, g] \rightarrow S^1$ is not continuous, then we can assume that the Latency L behaves in such a way that its independent at a point on stimulus timing and strength changing from weak to strong. In the heart's reaction to the various timing and stimulus, there are discontinuities in the reaction. As the Latency L is not continuous, there is a peculiar set of points $P \subset S^1 \times [w, g]$ that makes the Latency not continuous. Now, suppose we have one point of discontinuity of L which lies in $S^1 \times [w, g]$. Then, we can say that the boundary of K lies on the circle, where the latency function L takes on all values of latencies from 0 *to* 2π as seen in Figure 5.

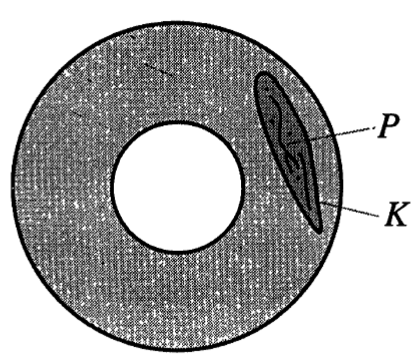

Figure 5: L is not continuous at P which is in K

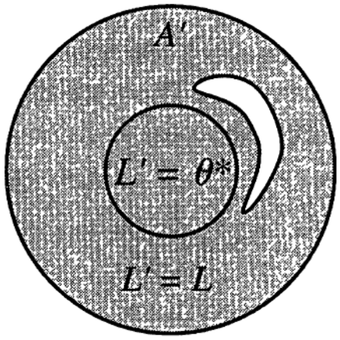

Figure 6: Extending L on A

We can see that the domain A' is topologically equivalent to the annulus. Then, when we restrict L'on the boundary of K, the degree of $L'|_{\delta K}$ is equal to the degree of $L'\ |s = w$. From our former discussion, the degree is −1. In Theorem 2, we know that a circle function which has nonzero degree is surjective.
Whence, assuming $L'|_{\delta K}$ is a function from the circle to itself, then it is surjective. We can conclude that, the boundary of K lies on the circle, where the Latency function L takes on all values of latencies from 0 to 2π. Now, around the set of discontinuities of Latency L, we can see that severe distribution of responses can occur. Considering P in L as a single point of discontinuity which lies inside $S^1 \times [w, g]$, we realize that the function L takes on all possible latency values on every circle surrounding P (Adams, C. and Franzosa, R., 2000).
Remarks 1: We can see there exist continuous function from S^1 to S^1. However, at the point P makes the function discontinuous. This causes fibrillation in the human heart.

2.2 Stereographic Projection
Theorem 3
Suppose $N = (0,0,1)$ is a point on the unit sphere which sits in the complex plane, then there exists a point $P = (x, y, 0)$ on the plane which relates to a unique point P' on the unit sphere (Welke, 1997), (Jangra, 2021).

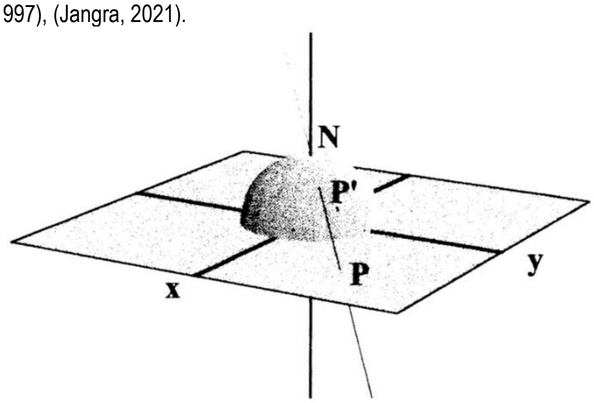

Figure 7: Stereographic Projection

Proof
The method is going from the plane to the sphere. Let $P = (x, y, 0)$ be a point in the complex plane and P' be the unique point on the unit sphere. Let $P' = P$ and
$N = (0,0,1)$ be a point on the north pole. Then the equation of the straight line NP is written as $\frac{X-x_1}{x_2-x_1} = \frac{Y-y_1}{y_2-y_1} = \frac{Z-z_1}{z_2-z_1}$

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = \lambda$$

$$\frac{X-0}{x-0} = \lambda, \qquad \frac{X}{x} = \lambda, \qquad X = x\lambda$$

$$\frac{Y-0}{y-0} = \lambda, \qquad \frac{Y}{y} = \lambda, \qquad Y = y\lambda$$

$$\frac{Z-1}{0-1} = \lambda, \qquad \frac{Z-1}{-1} = \lambda, \qquad Z = 1-\lambda$$

The coordinates of $NP = (x\lambda, y\lambda, 1-\lambda)$. Now, lets look for the value of λ which satisfy the unit sphere $X^2 + Y^2 + Z^2 = 1$.

$$(x\lambda)^2 + (y\lambda)^2 + (1-\lambda)^2 = 1$$
$$x^2\lambda^2 + y^2\lambda^2 + 1 - 2\lambda + \lambda^2 = 1$$
$$(x^2 + y^2 + 1)\lambda^2 + 1 - 2\lambda = 1$$
$$(x^2 + y^2 + 1)\lambda^2 - 2\lambda = 0$$
$$\lambda[(x^2 + y^2 + 1)\lambda - 2] = 0$$

But P is on the complex plane $(x + iy)$ and $|P|^2 = x^2 + y^2$
$\lambda[(x^2 + y^2 + 1)\lambda - 2] = 0, \lambda = 0$ or $\lambda = \frac{2}{|P|^2+1}$
When $\lambda = 0$ the point corresponds to $N = (0,0,1)$ because it is a point on the line and on the unit sphere. Thus, we take $\lambda = \frac{2}{|P|^2+1}$.
$X = \frac{2x}{|P|^2+1}$, where x is the real part
$Y = \frac{2y}{|P|^2+1}$, where y is the imaginary part
$Z = 1 - \lambda, Z = 1 - \frac{2}{|P|^2+1}, Z = \frac{|P|^2-1}{|P|^2+1}$
Therefore, the point $P = x + iy$ on the complex plane has corresponding points on the sphere: the stereographic projection $X = \frac{2x}{|P|^2+1}$, $Y = \frac{2y}{|P|^2+1}$ and $Z = \frac{|P|^2-1}{|P|^2+1}$
$P'(X, Y, Z)$.

3 RESULTS AND DISCUSSION
3.1 Numerical Experiments
Example 1

Let $P = 2 + 2i$, $x = 2$ is the real part and $y = 2$ is the imaginary part
$|P|^2 = 2^2 + 2^2 = 8$
$X = \frac{2x}{|P|^2+1} = \frac{2(2)}{8+1} = \frac{4}{9}$
$Y = \frac{2y}{|P|^2+1} = \frac{2(2)}{8+1} = \frac{4}{9}$
$Z = \frac{|P|^2-1}{|P|^2+1} = \frac{8-1}{8+1} = \frac{7}{9}$
$P'\left(\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right)$

Example 2

Let $P = 1 + \sqrt{3}i$, $x = 1$ is the real part and $y = \sqrt{3}$ is the imaginary part
$|P|^2 = 1^2 + \left(\sqrt{3}\right)^2 = 1 + 3 = 4$
$X = \frac{2x}{|P|^2+1} = \frac{2(1)}{4+1} = \frac{2}{5}$
$Y = \frac{2y}{|P|^2+1} = \frac{2(\sqrt{3})}{4+1} = \frac{2\sqrt{3}}{5}$
$Z = \frac{|P|^2-1}{|P|^2+1} = \frac{4-1}{4+1} = \frac{3}{5}$
$P'\left(\frac{2}{5}, \frac{2\sqrt{3}}{5}, \frac{3}{5}\right)$

Example 3

Let $P = 5 + 2i, x = 5$ is the real part and $y = 2$ is the imaginary part

$$|P|^2 = 5^2 + 2^2 = 25 + 4 = 29$$
$$X = \frac{2x}{|P|^2+1} = \frac{2(5)}{29+1} = \frac{10}{30} = \frac{1}{3}$$
$$Y = \frac{2y}{|P|^2+1} = \frac{2(2)}{29+1} = \frac{4}{30} = \frac{2}{15}$$
$$Z = \frac{|P|^2-1}{|P|^2+1} = \frac{29-1}{29+1} = \frac{28}{30} = \frac{14}{15}$$
$$P'\left(\frac{1}{3}, \frac{2}{15}, \frac{14}{15}\right)$$

Again, let us look at the reverse, thus going from the sphere to the complex plane. A point on the sphere is in the form (x_1, y_1, z_1) and a point on the plane is in the form $(x, y, 0)$. The points on the straight line is in the form $(x\lambda, y\lambda, 1 - \lambda)$. Then let us look for the value of λ such that we get (x_1, y_1, z_1).

$x_1 = x\lambda, y_1 = y\lambda$ and $z_1 = 1 - \lambda, \lambda = 1 - z_1$

$$x = \frac{x_1}{\lambda} = \frac{x_1}{1-z_1}$$
$$y = \frac{y_1}{\lambda} = \frac{y_1}{1-z_1}$$

Then, $\left(\frac{x_1}{1-z_1}, \frac{y_1}{1-z_1}, 0\right)$

When we start on the sphere (x_1, y_1, z_1) it corresponds to the point on plane

$p = \left(\frac{x_1}{1-z_1}, i\frac{y_1}{1-z_1}\right)$.

Remarks 2: But when we start on the north pole N on the sphere it corresponds to the point on the plane $p = \left(\frac{0}{1-z_1}, i\frac{0}{1-z_1}\right)$. This is not sensible so it means it runs to infinity.

Application: In astronomy, the stereographic projection is used to study the celestial objects from the earth.

4 CONCLUSION

The sphere is a powerful topological space. This paper demonstrated the real-life application of the one-sphere. *The human heart is assumed to be topologically equivalent to the one-sphere. When the human heart undergoes a stimulus, the time in the beat cycle is mapped to the time it recovers from the stimulus. Stereographic projection was computed successfully and used by astronomists. Further research should be done on applications of stereographic projection in other fields.*

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