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Full Length Research Article

AN INVESTIGATION INTO HOMOTOPY OF CONTINUOUS FUNCTIONS

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ABSTRACT
A homotopy is a continuous one-parameter family of continuous functions. This enquiry sought to find out how the various forms ranging from paths, inverses, reflexivity, symmetry and transitivity and other instances could be given in descriptive survey.

Keywords and phrases: path, loops, homotopy, continuous deformation, topological space, continuity, homotopy class.

INTRODUCTION

1.1 Preamble

Mathematically, Hurewicz will best be remembered for his important contributions to dimension, and above all as the founder of homotopy group theory. Suffice it to say that the investigation of these groups dominates present day topology (Lefschetz, 1957). One of the outstanding problems in homotopy theory is that of determining the homotopy groups of simple spaces. Even for as simple a space as the n-sphere very little is known. In fact, in most cases, it is not known whether or not the homotopy groups are zero (Whitehead, 1946). Two continuous functions from one topological space to another are called homotopic if one can be "continuously deformed" into the other, such a deformation being called a homotopy between the two functions (Suciu, 2008). A homotopy is a continuous one-parameter family of continuous functions from X to Y. If t denotes the parameter, the homotopy describes a continuous "deformation" of the function f into the function g as t increases from 0 to 1. The question of whether f is homotopic to g is a question of whether there is a continuous extension of a given function. We think of f as being a function from $X \times \{0\}$ into Y and g as being a function from $X \times \{1\}$ into Y, so we have a continuous function from $X \times \{0,1\}$ into Y and we want to extend it to a continuous function from $X \times I$ into Y (Armstrong, 1983). For any topological space which is metric, compact (hence separable) path connected and locally path connected, its homotopy group is not the additive group of the rational, moreover if it is not finitely generated then it has the cardinality of the continuum (Shelah, 1988). The ageing of the body is described by continuity and connectivity which are topological properties (Brew, Obeng-Denteh, & Zigli, 2019). They introduced a topological computable invariant: the homotopy to describe the process of ageing of human body. Homotopy offers a variety of ways which can be applied to myriad of systems as showed in research conducted by Issaka et al, (2016, 2017) on Fredholm integral equations using homotopy analysis. The basic pattern of classification, and at the same time the guarantee for its stability, is typically homotopy theory. That is, we consider two systems to be equivalent, or "in the same topological phase", if one can be deformed continuously into the other while retaining some key properties (Cedzich, et al., 2018). Based on the concept of the homotopy, computation methods for algebraic and differential equations have been developed. The method for algebraic equations includes homotopy continuation method (Agyekum, 2017).

1.2 Basic Definitions

1.2.1 A path in a topological space $\boldsymbol{X}$ is a continuous function $\boldsymbol{f}$ from the closed unit interval $\boldsymbol{I} = [\mathbf{0}, \mathbf{1}]$ into $\boldsymbol{X}$. The points $\boldsymbol{f}(\mathbf{0})$ and $\boldsymbol{f}(\mathbf{1})$ are the initial point and terminal point of $\boldsymbol{f}$ respectively.

Paths f and g with common initial point $f(0) = g(0)$ and common terminal point $f(1) = g(1)$ are equivalent provided that there is a continuous function $H: I \times I \to X$ such

$$H(t,0) = f(t), \quad H(t,1) = g(t), \quad t \in I,$$
$$H(0,s) = f(0) = g(0), \quad H(1,s) = f(1) = g(1), \quad s \in I.$$

The function H is called a **homotopy** between f and g.

1.2.2 Inverse Path

Given a path f in a topological space X, the **inverse path** of f is $f^{-1}(s) = f(1-s)$ (Dooley, 2011). This is a path that moves in the opposite direction of the original path f. For example, if $f(s)$ starts from x to y then $f^{-1}(s)$ will start from y to x.

1.2.3 Homotopy

Let X be a topological space with two paths $f(0)$ and $f(1)$ that have endpoints $x, y \in X$. A homotopy from $f(0)$ to $f(1)$ is a family of paths $f_t: [0,1] \to X$
such that for all $f(0)$ and $t \in [0,1]$, f_t satisfies the following conditions:

1. $f_t(0) = x$ and $f_t(1) = y$
2. The map $F: [0,1] \times [0,1] \to X$ defined by $F(s,t) = f_t(s)$ is continuous.

When there exists a homotopy between the two paths f_0 and f_1, then, the two paths are said to be homotopic. The homotopic relation between two paths is represented as $f_0 \simeq f_1$ (which is read as f_0 is homotopic to f_1).

The **homotopy class** of f, denoted $[f]$, is the equivalence class of a path f under the **equivalence relation** of homotopy (Dooley, 2011).

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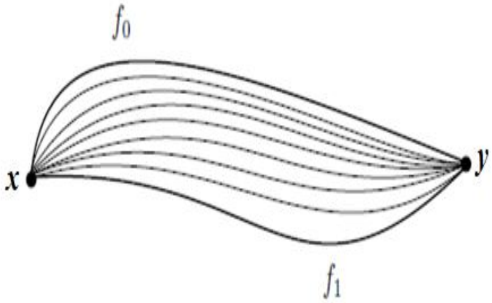

Figure 1: A path homotopy

1.2.4 Homotopy Equivalence

Two spaces X and Y are homotopy-equivalent: (or of the same homotopy type) if there exist maps $f: X \to Y$ and $g: X \to X$ such that $gf \simeq 1_x$ and $fg \simeq 1_y$, where 1_x and 1_y are the identity maps of X and Y respectively. In this case f is equivalence and g is a homotopy inverse to f. We write $X \simeq Y$ for 'X is homotopy-equivalent to Y' (notice that the symbol has two distinct meanings, depending on the context) (Maunder, 1996).

1.2.5 The operation · can be applied to homotopy classes as well. Consequently, let $\boldsymbol{f}: \boldsymbol{I} \to \boldsymbol{X}$ be a path from $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$ and let $\boldsymbol{g}: \boldsymbol{I} \to \boldsymbol{X}$ be a path from $\boldsymbol{x_1}$ to $\boldsymbol{x_2}$.
Define $[f] \cdot [g] = [f \cdot g]$ (Dooley, 2011).

1.2.6 Given two paths $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{I} \to \boldsymbol{X}$ such that $\boldsymbol{f(1)} = \boldsymbol{g(0)}$, there is a composition or product path $\boldsymbol{f} \cdot \boldsymbol{g}$ that traverses (travel across) first $\boldsymbol{f}$ and then $\boldsymbol{g}$, defined by the formula

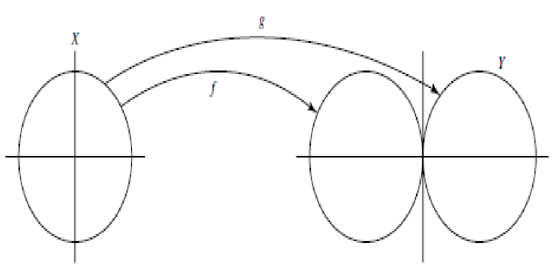

Figure 2

$$f \cdot g = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

This means that f and g are traversed twice as fast in order for $f \cdot g$ to be traversed in unit time. This **product operation** respects **homotopy classes** since if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via homotopies f_t and g_t, and if $f_0(1) = g_0(0)$ so that $f_0 \cdot g_0$ is defined, then $f_t \cdot g_t$ is defined and provides a homotopy
$f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

1.2.7 Given a topological space $\boldsymbol{X}$, a path $\boldsymbol{f}: \boldsymbol{I} \to \boldsymbol{X}$ is a loop if $\boldsymbol{f(0)} = \boldsymbol{f(1)} = \boldsymbol{x_0}$ for some $\boldsymbol{x_0} \in \boldsymbol{X}$.
The common value of the initial point and terminal point is referred to as the base point of the loop. Two loops f and g having common base point x_0 are equivalent or homotopic modulo x_0 provided that they are equivalent as paths. In other words, f and g are homotopic modulo x_0 (denoted $f \simeq_{x_0} g$) provided that there is a homotopy $H: I \times I \to X$ such that

$$H(\cdot, 0) = f \qquad H(\cdot, 1) = g \qquad H(0, s) = H(1, s) = x_0 \qquad s \in I$$

Since $H(0, s)$ and $H(1, s)$ always have value x_0 regardless of the choice of s
in $[0, 1]$, it is sometimes said that the base point "stays fixed throughout
the homotopy."
In this case the common starting and ending point, x_0, is called the basepoint.
The set of all **homotopy classes** $[\boldsymbol{f}]$ of loops $f: I \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

1.3 Examples of Homotopies

1.3.1 Let $\boldsymbol{X} = \mathbb{R}^{\boldsymbol{n}}$, the paths $\boldsymbol{f_0}, \boldsymbol{f_1}: \boldsymbol{I} \to \mathbb{R}^{\boldsymbol{n}}$ with the same endpoint (i.e., $\boldsymbol{f_0(0)} = \boldsymbol{f_1(0)} = \boldsymbol{x}$, and $\boldsymbol{f_0(1)} = \boldsymbol{f_1(1)} = \boldsymbol{y}$) are homotopic via the linear homotopy $\boldsymbol{f_t(s)} = \boldsymbol{(1-s)f_0(s)} + \boldsymbol{tf_1(s)}$. This shows that during this homotopy each point $\boldsymbol{f_0(s)}$ travels along the line segment to $\boldsymbol{f_1(s)}$ at constant speed.

1.3.2 For a convex set $\boldsymbol{X}$ in $\mathbb{R}^{\boldsymbol{n}}$ with basepoint $\boldsymbol{x_0} \in \boldsymbol{X}$ we have $\boldsymbol{\pi_1(X, x_0)} = \boldsymbol{0}$, the trivial group, since any two loops $\boldsymbol{f_0}$ and $\boldsymbol{f_1}$ based at $\boldsymbol{x_0}$ are homotopic via the linear homotopy $\boldsymbol{f_t(s)} = \boldsymbol{(1-s)f_0(s)} + \boldsymbol{tf_1(s)}$, as observed in Example 1 **(Hatcher, 2001)**.

1.3.3 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be subspaces of $\mathbb{R} \times \mathbb{R}$ defined by $\boldsymbol{X} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}: \boldsymbol{x^2} + \boldsymbol{y^2} = \boldsymbol{1}\}$ and $\boldsymbol{Y} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}: (\boldsymbol{x}+\boldsymbol{1})^2 + \boldsymbol{y^2} = \boldsymbol{1} \ \boldsymbol{or} \ (\boldsymbol{x}-\boldsymbol{1})^2 + \boldsymbol{y^2} = \boldsymbol{1}\}$. Define $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ by $\boldsymbol{f(x, y)} = (\boldsymbol{x}-\boldsymbol{1}, \boldsymbol{y})$ and $\boldsymbol{g(x, y)} = (\boldsymbol{x}+\boldsymbol{1}, \boldsymbol{y})$.

Then f is not homotopic to g.
Example 3 can be found in (Armstrong, 1983).

1.3.4 Let $\boldsymbol{X}$ be the subspace of $\mathbb{R} \times \mathbb{R}$ defined by $\boldsymbol{X} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}: \boldsymbol{x^2} + \boldsymbol{y^2} = \boldsymbol{1}\}$ and let $\boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{X}$. Define $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ by $\boldsymbol{f(x, y)} = ((\boldsymbol{1}, \boldsymbol{0}), (\boldsymbol{x}, \boldsymbol{y}))$ and $\boldsymbol{g(x, y)} = ((\boldsymbol{0}, \boldsymbol{1}), (\boldsymbol{x}, \boldsymbol{y}))$. Then the function $\boldsymbol{H}: \boldsymbol{X} \times \boldsymbol{I} \to \boldsymbol{Y}$ defined by $\boldsymbol{H}((\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{t}) = ((\sqrt{\boldsymbol{1}-\boldsymbol{t^2}}, \boldsymbol{t}), (\boldsymbol{x}, \boldsymbol{y}))$ is a homotopy between $\boldsymbol{f}$ and $\boldsymbol{g}$ so $\boldsymbol{f} \simeq \boldsymbol{g}$.

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Notice that X is a circle and Y is a torus, f "wraps" the circle X around $\{(1,0)\} \times X$, and g "wraps" the circle X around $\{(0,1)\} \times X$ (See Figure 3). Let A denote the arc from $((1,0),(1,0))$ to $((0,1),(0,1))$ (See Figure 3). Then H

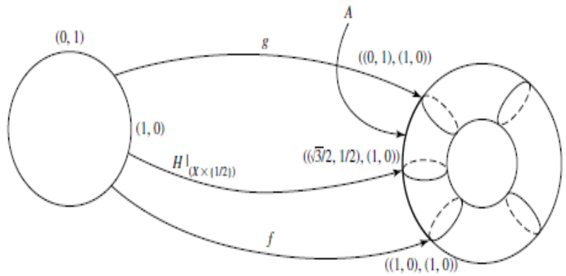

Figure 3

maps $X \times I$ onto $A \times X$.
Refer to (Armstrong, 1983) for more on Example 4.

1.3.5 Let $\boldsymbol{X = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}}$ (see Figure 1.2.3), let $\boldsymbol{\alpha}: \boldsymbol{I} \to \boldsymbol{X}$ be the path that maps $\boldsymbol{I}$ in a "linear" fashion onto the arc $\boldsymbol{A}$ from $(-\mathbf{1},\mathbf{0})$ to $(\mathbf{1},\mathbf{0})$ and let $\boldsymbol{\beta}: \boldsymbol{I} \to \boldsymbol{X}$ be the path that maps $\boldsymbol{I}$ in a "linear" fashion onto the arc $\boldsymbol{B}$ from $(-\mathbf{1},\mathbf{0})$ to $(\mathbf{1},\mathbf{0})$. Then $\boldsymbol{\alpha}$ is homotopic to $\boldsymbol{\beta}$ (by a homotopy that transforms $\boldsymbol{\alpha}$ into $\boldsymbol{\beta}$ in the manner illustrated in Figure), but $\boldsymbol{\alpha}$ is not path homotopic to $\boldsymbol{\beta}$ because we cannot "get from" $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ and keep the "endpoints" fixed without "crossing the hole" in the space **(Armstrong, 1983)**.

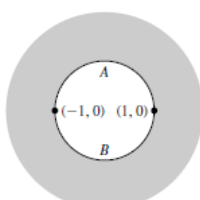

Figure 4

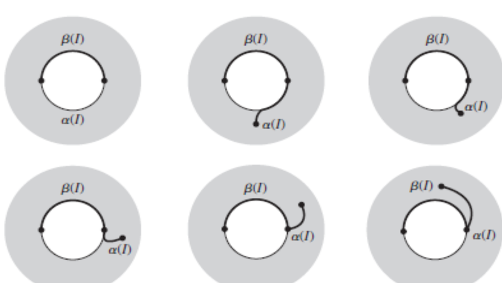

Figure 5

PROCEDURAL STRUCTURE

1.4 Basic theorems

1.4.1 Lemma 1

Let X be a topological space and A, B be closed subsets of X such that $X = A \cup B$. Let Y be a topological space and $f: A \to Y$ and $g: B \to Y$ be continuous maps. If $f(x) = g(x), \forall x \in A \cap B$, then the function $h: X \to Y$ defined by

$$h(x) := \begin{cases} f(x), & \forall x \in A \\ g(x), & \forall x \in B \end{cases}$$

is continuous.

Proof: From the above lemma, h is the unique well-defined function $X \to Y$ such that $h|_A = f$ and $h|_B = g$. Now we need to show that h is continuous. Let γ be a closed set in Y, then

$$\begin{aligned} h^{-1}(\gamma) = X \cap h^{-1}(\gamma) &= (A \cup B) \cap h^{-1}(\gamma) \\ &= \left(A \cap h^{-1}(\gamma)\right) \cup \left(B \cap h^{-1}(\gamma)\right) \\ &= \left(A \cap f^{-1}(\gamma)\right) \cup \left(B \cap g^{-1}(\gamma)\right) \\ &= \left(f^{-1}(\gamma)\right) \cup \left(g^{-1}(\gamma)\right) \end{aligned}$$

Since each of f and g is continuous, $f^{-1}(\gamma)$ and $g^{-1}(\gamma)$ are both closed in X. This implies that $h^{-1}(\gamma)$ is closed in X. Hence, h is continuous.
Refer to (Massey, 1991) and (Dooley, 2011) for the complete proof.

1.4.2 Proposition 1

Given a topological space X with two endpoints $x, y \in X$, path homotopy is an equivalence relation on the set of all paths from x to y.

Proof: To show that $\simeq$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. Let X be a topological space and consider some $x, y \in X$. Let f, g and h be paths from x to y. Then for reflexivity, it is obvious that $f \simeq f$ by the constant homotopy $f_t = f$ or the identity homotopy

$$f_t(s,t) = f(s) \qquad for\ all\ \ t \in I.$$

Now for symmetry, if $f \simeq g$, then we have a continuous $h(s,t)$ satisfying

$$h(s,0) = f(s), \qquad h(s,1) = g(s), \qquad h(0,t) = f(0), \qquad h(1,t) = f(1)$$

Define h inverse as $h_{1-t}(s,t) = h(s, 1-t)$. Then h_{1-t} is continuous because it is a composition of continuous maps. Also,

$$h_{1-t}(s,0) = h(s,1) = g(s)$$

$$h_{1-t}(s,1) = h(s,0) = f(s)$$

$$h_{1-t}(0,t) = f(0) = g(0)$$

$$h_{1-t}(1,t) = f(1) = g(1)$$

Hence, we have a homotopy $g \simeq f$.

Finally for transitivity, assume that $f \simeq g$ via a homotopy f_t and $g \simeq h$ via a homotopy g_t. Then we can see that $f \simeq h$ via the homotopy h_t that is defined by f_{2t} on $[0, \frac{1}{2}]$ and g_{2t-1} on $[\frac{1}{2}, 1]$. It is clear that the associated map $H(s,t)$ is continuous since, by assumption, it is continuous when restricted to the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and it agrees at $t = \frac{1}{2}$.

Refer to (Dooley, 2011) and (Hatcher, 2001) for more on Proposition 1.

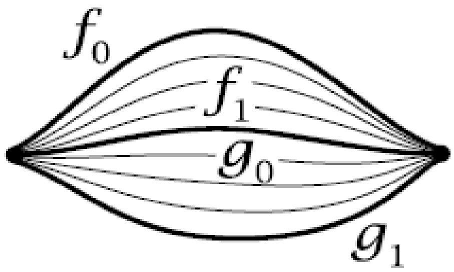

Figure 6: Equivalence relation of paths

1.4.3 Proposition 2

The operation $\cdot$ between homotopy classes is well-defined.

Proof: Let $f' \in [f]$ and $g' \in [g]$. Because $[f'] = [f]$ and $[g'] = [g]$, we want to check $[f'] \cdot [g'] = [f] \cdot [g]$. Because f and f' are path homotopic, there exists a path homotopy F from f to f'. Likewise, there exists a path homotopy G from g to g'.

$$H(s,t) = \begin{cases} F(2s,t) & s \in [\,0, \frac{1}{2}] \\ G(2s-1,t) & s \in [\frac{1}{2}, 1] \end{cases}$$

Now, we show that H is a path homotopy between $f \cdot g$ and $f' \cdot g'$, which are paths from x_0 to x_2. We know H is continuous by **Lemma 1**. For all $t \in I$, we have $H(0,t) = F(0,t) = x_0$ and $H(1,t) = G(1,t) = x_2$. For all $s \in I$, we have $H(s,0) = (f \cdot g)(s)$ and $H(s,1) = (f' \cdot g')(s)$. The function H is therefore a path homotopy between $f \cdot g$ and $f' \cdot g'$. Thus, $[f' \cdot g'] = [f \cdot g]$. This means $[f'] \cdot [g'] = [f] \cdot [g]$. Therefore $\cdot$ is well-defined (Munkres, 1996).

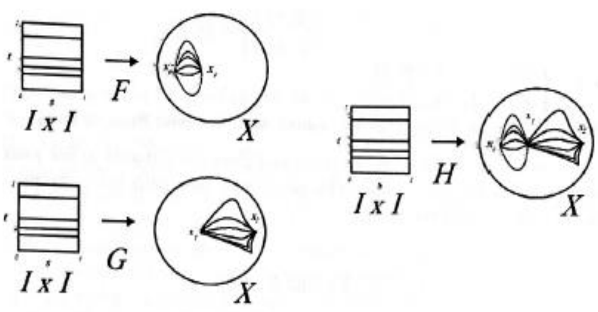

Figure 7: $[f] \cdot [g]$ is well-defined (Dooley, 2011).

1.4.4 Theorem 1

Given a topological space X, the set of homotopy classes $[f]$ of loops $f: I \to X$ at the basepoint x_0 forms a group under the product $[f] \cdot [g] = [f \cdot g]$. (i.e., $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$). We call $\pi_1(X, x_0)$ the **fundamental group**.

Proof: To prove that $\pi_1(X, x_0)$ is a group, we must show that $\pi_1(X, x_0)$ satisfies the group axioms:

1. Closure: We define the product of two classes $[f] \in (X, x_0)$ and $[g] \in (X, x_0)$ by $[f][g] = [f \cdot g]$. The definition of product is independent of the choice of representatives of $[f]$ and $[g]$. Because if $f \simeq f_1$ and $g \simeq g_1$ then $f \cdot g \simeq f_1 \cdot g_1$ and so $[f_1][g_1] = [f_1 \cdot g_1] = [f \cdot g]$. So, the product $[f][g]$ is uniquely defined by $[f]$ and $[g]$.This verifies the closure property of groups. We now verify that all the group axioms are satisfied.

2. Associativity: Given paths f, g, and h with $f(1) = g(0)$ and $g(1) = h(0)$, we define
$$([f][g])[h] = [f \cdot g][h] = [(f \cdot g).h]$$
and
$$[f]([g][h]) = [f][g \cdot h] = [f \cdot (g \cdot h)]$$

   From this, we can see that $f \cdot (g \cdot h)$ is a reparameterization of $(f \cdot g) \cdot h$ which is given by the function
$$f \cdot (g \cdot h) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ h(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$$
   From this, it is clear that
$$(f \cdot g) \cdot h = \begin{cases} f(4s), & 0 \le s \le \frac{1}{2} \\ g(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ h(2s-1), & \frac{3}{4} \le s \le 1 \end{cases}$$
   It is obvious that
$$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$$
   and is given by
$$h(s,t) = \begin{cases} f(2s)t + (4s)(1-t), & 0 \le s \le \frac{t}{2} + \frac{1-t}{2} \\ g(4s-2), & \frac{t}{2} + \frac{1-t}{4} \le s \le \frac{3t}{4} + \frac{1-t}{2} \\ h(4s-3)t + (2s-1)(1-t), & \frac{3}{4} + \frac{1-t}{2} \le s \le 1 \end{cases}$$

   Hence, the operation is associative.

3. There exists a constant path called the identity $e_{x_0} \in \pi_1(X, x_0)$ such that $[f] \cdot e_{x_0} = e_{x_0} \cdot [f] = [f]$: Let $e_{x_0}(s) = f(1) = x_0$ for all $s \in I$. Now, $f \cdot e_{x_0} \simeq f$ via the homotopy,
$$H(s,t) = \begin{cases} f((2-t)s), & 0 \le t \le \frac{1-t}{2} \\ e_{x_0}, & \frac{1-t}{2} \le t \le 1 \end{cases}$$

$$e_{x_0} \simeq f$$

hence, the identity exists.

4. For all $[f] \in \pi_1(X, x_0)$, there exists $[f]^{-1}$ such that

$$[f] \cdot [f]^{-1} = [f]^{-1} \cdot [f] = e_{x_0}$$

Take a representative of $[f]$ and call it f. Let $[f]^{-1}(s) = f(1-s)$ for all $s \in I$.
We need to show that $f \cdot [f]^{-1} \simeq e_{x_0}$.
Hence, we define

$$h(s,t) = \begin{cases} f(2s), & 0 \le s \le \frac{1-t}{2} \\ f\left(\frac{1-t}{2}\right), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ f^{-1}(s), & \frac{1+t}{2} \le s \le 1 \end{cases}$$

This means that $f \cdot [f]^{-1} \simeq e_{x_0}$ and $f^{-1} \cdot f \simeq e_{x_0}$ are similar (Dooley, 2011).

Conclusion

This enquiry sought to find out how the various forms ranging from paths, inverses, reflexivity, symmetry and transitivity culminating in equivalence relation and engaged in a descriptive survey. The survey has shown that not all continuous functions are actually homotopic just as seen discontinuous functions. The concept of homotopy is also been applied in other areas of mathematics; the use of the homotopy analysis method in providing solutions to Fredholm's Integral Equations of the Second Kind as seen in the work of (Issaka I. , Obeng-Denteh, Mensah, & Owusu-Mensah, 2016) and the application of homotopy to the ageing process of the human body (Brew, Obeng-Denteh, & Zigli, 2019).

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