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SUMUDU-BERNSTEIN SOLUTION OF DIFFERENTIAL, INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT
A numerical method based on the inverse Sumudu transform and the Bernstein polynomials operational matrix of integration is developed. The derived method is implemented in solving linear differential, integral and integro-differential equations. Also, a procedure for overcoming nonlinearity is developed and implemented to solve nonlinear Volterra integral equations. The approximate results are compared with the exact solutions and an existing method. Error estimation shows that the proposed method has elevated level of accuracy for just a few terms of the polynomial.

Keywords: Inverse Sumudu transform, Orthonormal Bernstein Operational matrix of integration, Differential equations, Integral equations, Inetgro-differential equations.

INTRODUCTION

Watugala (1993) introduced the Sumudu transform as a new integral transform and applied it to solve differential equations and control engineering problems. Further work by Weerakoon (1998) provided the complex inversion formula for the Sumudu transform and its properties like differentiation, integration, convolution theorems, shifting theorems, recurrence results and a comprehensive list of Sumudu transform of functions (Belgacem & Karablli, 2006). More properties of the Sumudu transform alongside applications to some dynamic partial differential equations problems arising in physics and engineering were provided by Kaya & Yilmaz (2019). Another important work on this subject was the tutorial based on utilizing a geometric Taylor series for finding the inverse of the Sumudu transform presented by Atlas *et al.* (2019). Recently, the Sumudu transform was applied to solve regular fractional continuous-time linear systems and fractional damped Burgers' equation approximately (Kaisserli & Bouagada, 2021).

A procedure for finding operational matrix of integration, differentiation and product for the Bernstein polynomials was introduced by Singh *et al.* (2009) and a general procedure of forming these matrices given by Yousefi & Beroozifar (2010). Also, Ordokhani & Far (2011) discussed the operational matrices of integration and product of the Bernstein polynomials and employed them to solve differential equations. The Bernstein polynomials method has also been used to solve parabolic equation, nonlinear Volterra-Fredhom-Hammerstein integral equations and systems of high order linear Volterra-Fredholm integro-differential equations (Maleknajad *et al.*, 2012a; Maleknajad *et a.l,* 2012b; Yousefi *et al.*, 2011).

An improvement on the procedure for finding operational matrix of integration, differentiation and product for the orthonormal Bernstein polynomials was sought by Bencheikh *et al.* (2016) while Javadi *et al.* (2016) applied the shifted orthogonal Bernstein polynomials to solve the generalized pantograph equations. To improve the accuracy and efficiency of the method, a modified work on the Bernstein polynomial operational matrix method was carried out and applied to solve Riccati differential equation and Volterra population model (Parand & Rad, 2016) and both linear and non-linear delay differential equations (Bataineh *et al.,* 2017).

Furthermore, the orthonormal Bernstein operation matrix of integration was used to investigate the intersect Laplace transform (Rani, 2018). This approach is based on replacing the unknown function through a truncated series of Bernstein basis polynomials while the coefficient of the expansion are obtained using the operational matrix of integration. The error and convergence analysis via residual function for this method was done by Bataineh (2018) while Rani *et al.* (2019) used this approach to find the numerical inverse Laplace transform for a class of fractional differential equation. More recently, Mishra & Rani (2020) worked on Laplace transform inversion using this method and applied it to solve differential and integral equations.

The focus of this paper is to use a combination of the inverse Sumudu transform and the orthonormal Bernstein polynomials matrix to compute approximate solutions of differential, integral and integro-differential equations.

MATERIALS AND METHODS

The Bernstein polynomial $b_n(t)$ of degree n (Bataineh *et al.,* 2017; Parand and Rad, 2016; Rani *et al.*, 2019) is defined as

$$b_n(t) = \sum_{i=0}^{n} b_{i,n}(t)\beta_i \tag{1}$$

where β_i's are the Bernstein coefficients and $b_{i,n}(t)$ is the Bernstein basis polynomial given by

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

where $\binom{n}{i}$ is a binomial coefficient. Since the Bernstein polynomials are not orthogonal, they are usually orthonormalised using the Gram-Schmidt orthonormalization procedure. We shall denote the orthonormal Bernstein polynomials as $B_{i,n}(t),\ i = 0,1,\ldots,n$. The n^{th} degree of the orthonormal Bernstein polynomial are defined on the interval [0,1] by

$$B_{i,n}(t) = \sqrt{2(n-i)+1}(1-t)^{n-i}\sum_{k=0}^{i}(-1)^k\binom{2n+1-k}{i-k}\binom{i}{k}t^{i-k} \tag{2}$$

for $i = 0,1,\ldots,n$ (Javadi *et al*, 2016). Equation (2) can be written implicitly as

$$B_{i,n}(t) = \sqrt{2(n-i)+1}(1t)^{n-i}\sum_{k=0}^{i}(-1)^k \frac{\binom{2n+1-k}{i-k}\binom{i}{k}}{\binom{n-k}{i-k}} B_{i-k,n-k}(t) \quad (3)$$

The orthonormal Bernstein polynomials satisfy the following relation

$$\int_0^1 B_{i,n}(x)B_{j,n}(x)dx = \delta_{i,j}, \quad i,j = 0,1,\ldots,n \quad (4)$$

where $\delta_{i,j}$ is the Kronecker delta function.
Equation (2) can also be expressed in matrix form as

$$B(t) = N\tau_n(t), \quad t \in [0,1] \quad (5)$$

where N is the matrix of constant coefficients

$$N_{i,j} = \sqrt{2(n-i)+1}\sum_{k=\max\{0,j-n+i\}}^{\min\{i,j\}} \alpha_{i,j-k}\beta_{i,k}, \quad i,j = 0,1,\ldots,n$$

and $\tau_n(t) = [1, t, t^2, \ldots, t^n]^T$.

The orthonormal Bernstein polynomials can be expressed in terms of the operational matrices of integration (Rani *et al.*, 2018). Let M_{n+1} be an $(n+1)\times(n+1)$ operational matrix of integration, then

$$\int_0^t B(x)dx \simeq M_{n+1}B(t), \quad t \in [0,1]$$

From (5), we have

$$\int_0^t B(x)dx = N\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{2} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{n+1}\end{bmatrix}\begin{bmatrix}t\\ t^2\\ \vdots\\ t^{n+1}\end{bmatrix} = N\wedge\tau$$

where $\wedge$ is an $(n+1)\times(n+1)$ matrix,

$$\wedge = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{2} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{n+1}\end{bmatrix} \text{ and } \tau = \begin{bmatrix}t\\ t^2\\ \cdots\\ t^{n+1}\end{bmatrix}.$$

We can now approximate the element of vector τ in terms of $\{B_{i,n}(t)\}_{i=0}^n$ by (5) to have $\tau_n(t) = N^{-1}B(t)$, then for $k = 0,1,\ldots,n$

$$t^k = N_{k+1}^{-1}B(t)$$

where N_{k+1}^{-1} is $(k+1)^{th}$ row of N_{k+1}^{-1} for $k = 0,1,\ldots,n$. That is

$$\begin{bmatrix}N_1^{-1}\\ N_2^{-1}\\ \vdots\\ N_{k+1}^{-1}\end{bmatrix}.$$

Approximating t^{n+1} using

$$f(t) = \sum_{i=0}^{n} c_{i,n}B_{i,n}(t)$$

we have

$$t^{n+1} = c_{n+1}^T B(t)$$

where

$$c_{n+1} = \int_0^1 x^{n+1}B(x)dx.$$

Let

$$A = \begin{bmatrix}N_2^{-1}\\ N_3^{-1}\\ \vdots\\ N_{k+1}^{-1}\\ c_{n+1}^T\end{bmatrix}$$

we get

$$\int_0^t B(x)dx \simeq N\wedge AB(t), \quad t\in[0,1]$$

and we have the operational matrix of integration as

$$M \simeq N\wedge A.$$

The Sumudu - Bernstein Approach

Consider the Sumudu transform

$$F(u) = S[f(t)] = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}f(t)dt. \quad (6)$$

The inverse Sumudu transform is defined as

$$f(t) = F^{-1}(u) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{t}{u}}F\left(\frac{1}{u}\right)\frac{du}{u}. \quad (7)$$

Now, $f(t)$ is considered to be square integrable on $[0,1]$ and can be expressed in terms of Bernstein polynomials as

$$f(t) \simeq \sum_{i=0}^{n} c_i B_{i,n}(t) = C^T B(t) \quad (8)$$

where $C = [c_0, c_1, c_2, \ldots, c_n]^T$ and $B(t) = [B_{0,n}, B_{1,n}, B_{2,n}, \ldots, B_{n,n}]^T$.

We shall apply this technique to a time varying differential equation

$$af''(t) + bf'(t) + f(t) = v(t), \quad f(0) = f'(0) = 0 \quad (9)$$

where $v(t)$ is the unit step function. Integrating both sides twice from 0 to t yields

$$af(t) + b\int_0^t f(x)dx + \int_0^t\int_0^t f(x)dxdx = \int_0^t\int_0^t v(x)dxdx. \quad (10)$$

Taking the Sumudu transform of (10), we get

$$aF(u) + buF(u) + u^2F(u) = u^2 \quad (a+bu+u^2)F(u) = u^2 \; F(u) = \frac{u^2}{a+bu+u^2}.$$

Replacing u with associated matrix of integration M_{n+1}, we get

$$\tilde{F}(M_{n+1}) = M_{n+1}^2(aI + bM_{n+1} + M_{n+1}^2)^{-1} \quad (11)$$

where I is the identity matrix. We can also express the solution

$$\tilde{f}(t) = C^T B(t)$$

by associated matrix of integration as

$$\int_0^t \tilde{f}(x)dx = C^T M_{n+1}B(t)$$

$$\int_0^t\int_0^t \tilde{f}(x)dxdx = C^T M_{n+1}^2 B(t).$$

Also

$$\int_0^t\int_0^t v(x)dxdx = d^T M_{n+1}^2 B(t)$$

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Equation (10) becomes

$$aC^T B(t) + bC^T M_{n+1} B(t) + C^T M_{n+1^2} B(t) = d^T M_{n+1}^2 B(t).$$

Simplifying, we get

$$C^T = d^T M_{n+1}^2 (aI + bM_{n+1} + M_{n+1}^2)^{-1}.$$

By equation (11), we get

$$C^T = d^T \tilde{F}(M_{n+1}). \quad (12)$$

Application to Differential and Integral Equations

The applicability of the proposed method shall be tested on some differential, integral and integro-differential equations. This shall be achieved using the Sumudu transform theorems for differentiation, integration and shifting (Mishra & Rani, 2020). However, to solve the nonlinear component, we shall use the following approach.

Consider the nonlinear Voltera integral equation of the first kind with convolution kernel given by

$$x(t) = \alpha \int_0^t K(t-s) N(f(s)) ds \quad (13)$$

where $K(t-s)$ is the kernel, $x(t)$ is the known function, $N(f(t))$ is the nonlinear term and $f(t)$ is the unknown function. Taking the Sumudu transform of (13)

$$S[x(t)] = \alpha S[K(t)] S[N(f(t))] \quad (14)$$

we can rewrite as

$$S[N(f(t))] = \frac{S[x(t)]}{\alpha S[K(t)]}$$

$$N(f(t)) = S^{-1}\left[\frac{S[x(t)]}{\alpha S[K(t)]}\right].$$

Thus, the approximate solution can be obtained by the proposed method as described above.

Furthermore, the error function could be estimated. Let $f_n(t)$ be the approximate solution and $r_n(t)$ be the perturbation function which depends on it. Consider

$$r_n(t) = \alpha \int_0^t K(t-s) N(f_n(s)) ds - x(t) \quad (15)$$

Substracting (15) from (13), we get

$$-r_n(t) = \alpha \int_0^t K(t-s) e_n(s) ds \quad (16)$$

We define the error function $e_n = N(f(t)) - N(f_n(t))$. Solving (16) using the same approach above, we get

$$S[e_n(t)] = \frac{-S[r_n(t)]}{\alpha S[K(t)]}. \quad (17)$$

Using the inverse Sumudu and the value of $r_n(t)$ in (15), we get

$$e_n(t) = S^{-1}\left[\frac{S[x(t)]}{\alpha S[K(t)]}\right]. \quad (18)$$

RESULTS AND DISCUSSION

Example 1: Find the solution of the linear differential equation

$$f^{iv}(t) + 2a^2 f''(t) + a^4 f(t) = \cos at$$

with initial conditions $f(0) = f'(0) = f''(0) = f'''(0) = 0$ and exact solution $f(t) = \frac{t\sin at - at^2 \cos at}{8a^3}$.

Solution: Taking the Sumudu transform, we get

$$\frac{1}{u^4} F(u) + \frac{2a^2}{u^4} F(u) + a^4 F(u) = \frac{1}{1 + a^2 u^2}$$

$$(1 + 2a^2 u^2 + a^4 u^4) F(u) = \frac{u^4}{1 + a^2 u^2}$$

$$F(u) = \frac{u^4}{(1 + a^2 u^2)^3}.$$

The operational matrix of integration is

$$\tilde{F}(M_{n+1}) = M_{n+1}^4 ((I + a^2 M_{n+1}^2)^3)^{-1}.$$

The coefficient matrix is

$$C^T = d^T M_{n+1}^4 ((I + a^2 M_{n+1}^2)^3)^{-1}.$$

For $a = 1$ and $n = 6$, we get

$$C^T = [0.0000638 \quad 0.0005236 \quad 0.001958 \quad 0.004468 \quad 0.007002 \quad 0.007842 \quad 0.005378]$$

and

$$d = [0.515079 \quad 0.473804 \quad 0.428571 \quad 0.377964 \quad 0.319438 \quad 0.247436 \quad 0.142857]^T.$$

The approximate solution is

$$\tilde{f}(t) = 0.0000001590240039 - 0.000008633022005\, t + 0.0001124634241\, t^2 - 0.0005968575313\, t^3 + 0.04320171335\, t^4 - 0.00196572208\, t^5 - 0.003097197322\, t^6.$$

Table 1: Absolute Errors for Example 1 at $n = 6$.

t	Exact: $f(t)$	Approximate: $\tilde{f}(t)$	Error: $\lvert f(t) - \tilde{f}(t) \rvert$
0	0	0.000000159024004	1.590240×10^{-7}
0.1	0.000004162501488	0.000004120915430	4.158605×10^{-8}
0.2	0.000066400380670	0.000066451585984	5.120531×10^{-8}
0.3	0.000334472247137	0.000334474988868	2.741731×10^{-9}
0.4	0.001049697235375	0.001049649828513	4.740686×10^{-8}
0.5	0.002539641103689	0.002539635738898	5.364790×10^{-9}
0.6	0.005208082833692	0.005208129479809	4.664612×10^{-8}
0.7	0.009522463662123	0.009522471151009	7.488886×10^{-9}
0.8	0.015999072342179	0.015999020424353	5.191783×10^{-8}
0.9	0.025186268045687	0.025186302793820	3.474813×10^{-8}
1.0	0.037646084867470	0.037645925843469	1.590240×10^{-7}

Example 2: Find the solution of the linear Volterra integral equation

$$\int_0^t \cos(t-s)f(s)ds = t\sin t$$

with the exact solution $f(t) = 2\sin t$.

Solution: Taking the Sumudu transform, we get

$$\frac{u}{1+u^2}F(u) = \frac{2u^2}{(1+u^2)^2}.$$

Simplifying, we get

$$F(u) = \frac{2u}{1+u^2}.$$

The operational matrix of integration is

$$\tilde{F}(M_{n+1}) = 2M_{n+1}(I + M_{n+1}^2)^{-1}.$$

The coefficient matrix is

$$C^T = d^T 2M_{n+1}(I + M_{n+1}^2)^{-1}.$$

For $n = 7$, we get the approximate solution as

$$\begin{aligned}\tilde{f}(t) = &-0.000000001832443488 + 2.000000129\,t\\ &-0.000002187209184\,t^2 - 0.3333178318\,t^3\\ &-0.00005556355224\,t^4 + 0.01677471441\,t^5\\ &-0.0001119356381\,t^6 - 0.0003453551998\,t^7.\end{aligned}$$

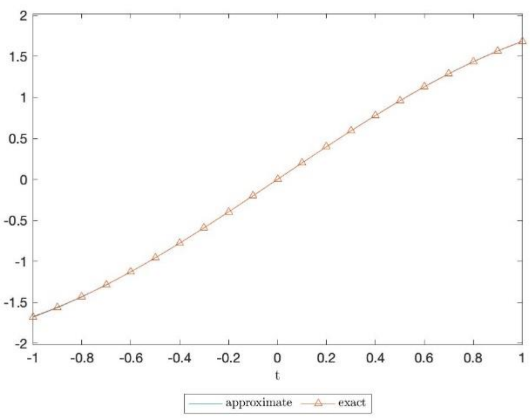

Figure 1: Exact and Approximate solution for Example 2.

Example 3: Find the solution of the linear integro-differential equation

$$f'(t) = 1 + \int_0^t f(s)ds$$

with the exact solution $f(t) = \sinh(t)$.

Solution: Taking the Sumudu transform, we get

$$\frac{1}{u}F(u) = 1 + uF(t).$$

Simplifying, we obtain

$$F(u) = \frac{u}{1-u^2}$$

The operational matrix of integration is

$$\tilde{F}(M_{n+1}) = M_{n+1}(I - M_{n+1}^2)^{-1}.$$

The coefficient matrix is

$$C^T = d^T 2M_{n+1}(I + M_{n+1}^2)^{-1}.$$

For $n = 8$, we obtain the following

$$\begin{aligned}\tilde{f}(t) = &\,0.0000000000643842 + 0.9999999942\,t\\ &+0.0000001261738432\,t^2 + 0.1666654955\,t^3\\ &+0.000005673420278\,t^4 + 0.008317569844\,t^5\\ &+0.00002600661051\,t^6 + 0.0001733083844\,t^7\\ &+0.00001301932268\,t^8.\end{aligned}$$

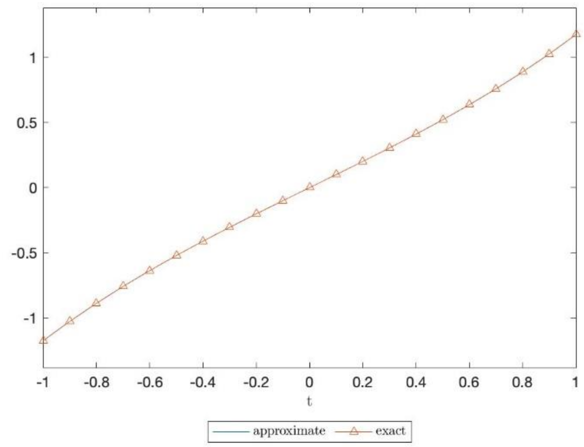

Figure 2: Exact and Approximate Solutions for Example 3

Example 4: Find the solution of the nonlinear integral equation

$$te^t = \int_0^t e^{(t-s)}e^{f(s)}ds$$

with the exact solution $f(t) = t$.

Solution: Taking the Sumudu transform, we get

$$\frac{u}{(1-u)^2} = \frac{u}{1-u}S[e^{f(t)}].$$

Simplifying, we get

$$S[e^{f(t)}] = \frac{1}{1-u}.$$

That is $e^{f(t)} = S^{-1}F(u)$ where $F(u) = \frac{1}{1-u}$. If we take $e^{f(t)} = X(t)$, the solution will be $\tilde{f}(t) = \ln X(t)$.

The operational matrix of integration is

$$\tilde{F}(M_{n+1}) = (I - M_{n+1})^{-1}.$$

The coefficient matrix is

$$C^T = d^T(I - M_{n+1})^{-1}.$$

For $n = 9$, we obtain the following

$$\begin{aligned}X(t) = &\,1.0 + 1.0\,t + 0.4999999928\,t^2\\ &+ 0.1666667492\,t^3\\ &+0.04166616695\,t^4 + 0.008335106664\,t^5\\ &+0.001385021617\,t^6 + 0.0002036372065\,t^7\\ &+0.00002058033641\,t^8 +\\ &0.000004573415464\,t^9.\end{aligned}$$

Therefore, the approximate solution is

$$\tilde{f}(t) = \ln X(t).$$

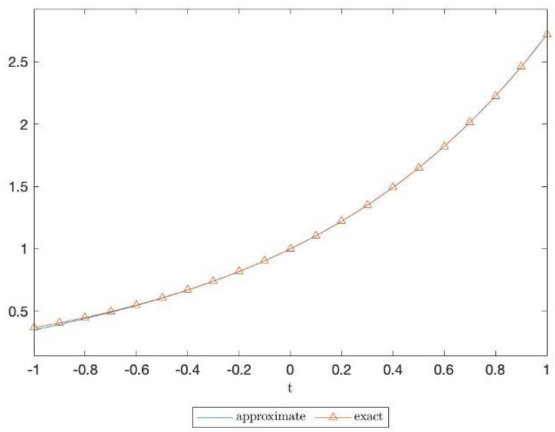

Figure 3: Exact and Approximate Solutions for Example 4.

Example 5: Find the solution of the nonlinear integral equation

$$\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 = \int_0^t (t-s)f^2(s)ds$$

with the exact solution $f(t) = 1 + t$.

Table 2: Absolute Errors for Example 5 at $n = 6$.

t	$f(t)$	Approximate: $\tilde{f}(t)$	Error: $\lvert f(t) - \tilde{f}(t)\rvert$
0	1	1	0
0.1	1.1	1.100000000000000	0
0.2	1.2	1.200000000000001	1.110223×10^{-15}
0.3	1.3	1.299999999999997	3.108625×10^{-15}
0.4	1.4	1.400000000000008	7.993606×10^{-15}
0.5	1.5	1.500000000000001	1.110223×10^{-15}
0.6	1.6	1.599999999999990	9.992007×10^{-15}
0.7	1.7	1.700000000000005	5.107026×10^{-15}
0.8	1.8	1.800000000000023	2.287059×10^{-14}
0.9	1.9	1.900000000000038	3.819167×10^{-14}
1.0	2.0	1.999999999999947	5.306866×10^{-14}

Table 3: Comparison of Relative Errors for Example 5 at $n = 6$.

t	Method in [20]	Proposed Method
0	2.4×10^{-6}	0
0.2	1.0×10^{-5}	1.1×10^{-15}
0.4	4.4×10^{-6}	5.6×10^{-15}
0.6	1.9×10^{-5}	6.3×10^{-15}
0.8	3.4×10^{-5}	1.3×10^{-14}
1.0	1.6×10^{-5}	2.6×10^{-14}

Example 1 is a fourth order linear differential equation which is transformed to a Sumudu transform with the aid of the Sumudu function differentiation theorem and then solved based on the proposed method when $a = 1$ and $n = 6$. The approximate solution is obtained with its absolute errors presented in Table 1. Example 2 is a linear Volterra integral equation of the first kind which is transformed to a Sumudu transform with the aid of the Sumudu function integration theorem and then solved at $n = 7$ and its graph with the exact solution plotted in Figure 1. Example 3 is a first order linear integro-differential equation which is

Solution: Taking the Sumudu transform, we get

$$\frac{1}{2}(2!\,u^2) + \frac{1}{3}(3!\,u^3) + \frac{1}{12}(4!\,u^4) = u^2 S[f^2(t)].$$

That is

$$u^2 + 2u^3 + 2u^4 = u^2 S[f^2(t)]$$
$$S[f^2(t)] = 1 + 2u + 2u^2.$$

Thus $f^2(t) = S^{-1}[F(u)]$, where $F(u) = 1 + 2u + 2u^2$. If we take $f^2(t) = X(t)$, then the solution will be $\tilde{f}(t) = \sqrt{X(t)}$.

The operational matrix of integration is

$$\tilde{F}(M_{n+1}) = I + 2M_{n+1} + 2M_{n+1}^2.$$

The coefficient matrix is

$$C^T = d^T(I + 2M_{n+1} + 2M_{n+1}^2).$$

For $n = 6$, we get

$$X(t) = 1 + 2\,t + \,t^2 + 2.1316282 \times 10^{-14}\,t^3 - 1.1901591 \times 10^{-13}\,t^4 - 4.7384319 \times 10^{-13}\,t^5 + 3.5482728 \times 10^{-13}\,t^6.$$

Therefore, the approximate solution is

$$\tilde{f}(t) = \sqrt{X(t)}$$

transformed to a Sumudu transform with the aid of the Sumudu function differentiation and integration theorems and then solved based on the proposed method when $n = 8$.
Example 4 is a nonlinear Volterra integral equation of the first kind whose linear part is tranformed into the Sumudu trasnform using the Sumudu exponential shifting theorem while the nonlinear component is transform using the approach presented in Section 4. The approximate solution is obtained at $n = 9$ and its graph with the exact solution is plotted in Figure 3. Similarly, Example 5 is a nonlinear Volterra integral equation whose approximate solution is obtained at $n = 6$ and its absolute errors presented in Table 2. Table 3 is a comparison of the relative errors between a Laplace transform based method presented in Mishra & Rani (2020) and the proposed method.

CONCLUSION

The Sumudu-Bernstein method was developed and applied to solve differential, integral and integro-differential equations. Numerical examples were presented including linear differential, intgeral, integro-differential and nonlinear integral equations after an approach to solve the nonlinear component was provided. The result of the absolute errors between the exact and the approximate solutions for Examples 1 and 5 respectively presented in Tables 1 and 2 and the graphs of Examples 2 to 4 respectively presented in Figures 1 to 3 shows that the proposed method gives elevated accuracy for just a few terms of the polynomial. A comparison of the propsed method with an existing method presented in Table 3 shows that the proposed method is more accurate. All computations were done using MATLAB 2021.

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