COMMUTATIVITY OF PRIME RINGS WITH MULTIPLICATIVE (GENERALIZED- REVERSED) DERIVATION

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ABSTRACT
Let R be a prime ring and F a multiplicative (generalized) reverse-derivative associated with mapping d on R. In this paper, we prove the commutativity of prime rings involving multiplicative (generalized) reverse- derivation. In addition, we prove that; for a prime ring , if $d(x)d(y) \pm xy = 0$ for all $x, y \in R$ then $d = 0$, where d is a skew- derivative associated with an automorphism $\beta: R \to R$.

Keywords: Prime ring, derivation, generalized derivation, multiplicative (generalized)-reverse derivation, skew derivation.

INTRODUCTION
In this paper, the symbol $[x, y]$ and (xoy), represent the Lie product $xy - yx$ and the Jordan product $xy + yx$, respectively, where $x, y \in R$. A ring R is called prime if $aRb = \{0\}$ for any $a, b \in R$, implies that $a = 0$ or $b = 0$ and is said to be semiprime if $aRa = 0$ for any $a \in R$, implies that $a = 0$.
Bresar & Vukman (1989) defined an additive mapping as follows: a mapping f is said to be an additive mapping on R if $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. According to Posner (1957) a mapping $d: R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. If d is an additive mapping then d is said to be a derivation on R. Also an additive mapping $F: R \to R$ is called generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. The notion of multiplicative derivation was first introduced by Daif (1997), according to him, a mapping $D: R \to R$ is called multiplicative derivation if it satisfies $D(xy) = D(x)y + xD(y)$, for all $x, y \in R$ where the mappings are not supposed to be an additive. Further Daif and Tammam El-sayiad (1997) extended multiplicative derivation to multiplicative generalized derivation, that is, a mapping F on R is said to be a multiplicative generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. From the definition of multiplicative generalized derivation if d is any mapping not necessarily additive then F is said to be multiplicative (generalized) derivation. Recently Dhara & Ali (2013) gave a more precise definition of multiplicative (generalized) derivation as follows: A mapping $F: R \to R$ is said to be a multiplicative (generalized) derivation if there exists a map g on R such that $F(xy) = F(x)y + xg(y)$, for all $x, y \in R$. Where g is any mapping on R (not necessarily additive). Therefore the concept of multiplicative (generalized) derivation cover the concept of multiplicative derivation and multiplicative generalized derivation.
The notion of reverse derivation was initiated by Herstein (1957). According to him, an additive mapping d on R is said to be reverse derivation if $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$. While

according to Abobakar & Gonzles (2015), the generalized reverse derivation is an additive mapping $F: R \to R$ if there exists a mapping $d: R \to R$ such that $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$. A mapping $F: R \to R$ is called multiplicative (generalized)-reverse derivation if $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$, where d is any mapping on R and F is not necessarily additive Tiwari et al., (2015).
In this present paper, we establish the commutativity of prime rings involving multiplicative
(generalized)-reverse derivation on R.

MAIN RESULTS
Theorem 1
Let R be a prime ring and F be a multiplicative (generalized) - derivative associated with mapping d on R. If $F(x)F(y) \pm xy = 0$ for all $x, y \in R$ then $d = 0$.

Proof. First we consider the case
$$F(x)F(y) + xy = 0 \tag{1}$$

for all $x, y, \in R$. Substituting yz instead of y in equation (1), we obtain
$$F(x)F(yz) + xyz = 0$$
$$F(x)(F(y)z + yd(z)) + xyz = 0$$
$$F(x)F(y)z + F(x)yd(z) + xyz = 0$$

But $F(x)F(y) = -xy$

Therefore, $-xyz + F(x) yd(z) + xyz = 0$
$$F(x)yd(z) + xyz - xyz = 0,$$
for all $x, y, z \in R \tag{2}$

Substituting xr instead of x in equation (2)
$$F(xr)yd(z) = 0$$
$$((F(x)r + xd(r))yd(z) = 0$$
$$F(x)ryd(z) + xd(r))yd(z) = 0 \ \forall \ x, y, z \in R \tag{3}$$

Substituting ry instead of y in equation (2), we obtain
$$F(x)ryd(z) = 0 \tag{4}$$

Subtracting equation (3) from equation (4), we obtain
$$xd(r)yd(z) = 0, \forall \ x, y, r, z \in R \tag{5}$$

Replacing $xd(r)$ by $d(t)$ in (5), we get
$$d(t)yd(z) = 0 \ \forall \ y, t, z \in R$$

Since d is mapping on $R\forall\ y,t,z \in R$.
This implies that, $d(t)Rd(z) = 0 \ \forall\ t,z \in R$

Therefore, by primeness of R, we obtain $d(t) = 0$ or $d(z) = 0$. Using similar approach we can prove that the same result holds for

$$F(x)F(y) - xy = 0 \forall\ x,y,r,z \in R.$$

Theorem 2
Let R be prime ring and F be a multiplicative (generalized) reverse- derivative associated with mapping d on R. If $F(x)F(y) \pm xy = 0$ for all $x,y \in R$ then R is commutative.

Proof: first we consider the case,
$$F(x)F(y) + xy = 0 \tag{6}$$

for all $x,y, \in R$. Substituting zy instead of y in equation (1), we obtain

$F(x)F(zy) + xy = 0$, for all $x,y,z \in R$.

By definition of generalized derivation, we have

$$F(x)\big(F(y)z + yd(z)\big) + xy = 0$$
$$F(x)\big(F(y)z + F(x)yd(z)\big) + xy = 0$$
But $F(x)F(y) = -xy$
$$-xyz + F(x)yd(z) + xzy = 0$$
$$xzy - xyz + F(x)yd(z) = 0$$
$$-x([z,y]) + F(x)yd(z) = 0$$
$$x[z,y] + F(x)yd(z) = 0$$

$$F(x)yd(z) + x[z,y] = 0 \tag{7}$$

for all $x,y, \in R$. Substituting ry instead of y in equation (7) where $r \in I$, we obtain

$$F(x)ryd(z) + x[z,ry] = 0 \ \forall\ x,y,r,z \in R$$
$$F(x)ryd(z) + x(r[z,y] + [z,r]y) = 0$$

$$F(x)ryd(z) + xr[z,y] + x[z,r]y = 0 \tag{8}$$

Replacing x instead of with rx in equation (7), we get

$$F(rx)yd(z) + rx[z,y] = 0 \tag{9}$$

By definition of generalized reverse derivation, we get

$$(F(x)r + xd(r)) + rx[z,y] = 0$$

$$F(x)ryd(z) + xd(r)yd(z) + rx[z,y] = 0 \tag{10}$$

Subtracting equation (8) by (10), we get
$$xr[z,y] + x[z,r]y - xd(r)yd(z) - rx[z,y] = 0$$

$$x[z,r]y - xd(r)yd(z) + [x,r][z,y] = 0 \tag{11}$$

for all $x,y,r,z \in R$. Substituting tx instead of x in equation (11), we obtain

$$tx[z,r]y - txd(r)yd(z) + [tx,r][z,y] = 0$$

$tx[z,r]y - txd(r)yd(z) + (t[x,r] + [t,r]x)[z,y] = 0$

$tx[z,r]y - txd(r)yd(z) + t[x,r][z,y] + [t,r]x[z,y] = 0 \tag{12}$

Multiplying equation (11) by t on the left side, we obtain

$$tx[z,r]y - txd(r)yd(z) + t[x,r][z,y] = 0 \tag{13}$$

Subtracting equation (11) by (12), we get
$$[x,r]x[z,y] = 0 \ \forall\ x,y,r,z \in R$$
$$[t,r]R[z,y] = 0 \ \forall\ t,y,r,z \in R$$

By primeness of R, then either $[t,r] = 0$ or $[z,y] = 0$
that is $tr - rt = 0$ or $zy - yz = 0$.

Therefore, $tr = rt$ or $zy = yz$ which implies that R is commutative.

Using similar approach we can prove that the same result holds for
$$F(x)F(y) - xy = 0 \forall\ x,y,r,z \in R.$$

Theorem 3
Let R be prime ring and d be a skew- derivative associated with an automorphism $\beta: R \to R$.

If $d(x)d(y) \pm xy = 0$ for all $x,y \in R$ then $d = 0$.

Proof: first we consider the case

$$d(x)d(y) + xy = 0 \tag{14}$$

for all $x,y, \in R$. Substituting yz instead of y in equation (14), we obtain
$$d(x)d(yz) + xyz = 0$$

By definition of skew derivation, we get

$$d(x)\big(d(y)z + \beta(y)d(z)\big) + xyz = 0 \ \forall\ x,y,z \in R$$
$$d(x)d(y)z + d(x)\beta(y)d(z) + xyz = 0$$
But $d(x)d(y) = -xy$
$$-xyz + d(x)\beta(y)d(z + xyz) = 0$$
$$d(x)\big(\beta(y)d(z)\big) + xyz - xyz = 0$$
$$d(x)\big(\beta(y)d(z)\big) = 0 \tag{15}$$

Replacing xr instead x in equation (15), we obtain
$$d(xr)\big(\beta(y)d(z)\big) = 0$$

By definition of skew derivation, we have
$$(d(x)r\big(\beta(x)d(r)\big)\beta(y)d(z) = 0$$

$$d(x)r\beta(y)d(z) + \beta(x)d(r)\beta(y)d(z) = 0 \tag{16}$$

Replacing $r\beta(y)$ instead of $\beta(y)$ in equation (15), we get

$$d(x)r\beta(y)d(z) = 0 \ \forall\ x.y,r,z \in R \tag{17}$$

Subtracting equation (17) equation (16), we obtain

$$\beta(x)d(r)\beta(y)d(z) = 0 \ \forall\ x.y,r,z \in R \tag{18}$$

Replacing $d(r)$ instead of $\beta(x)d(r)$ in equation (18), we get
$$d(r)\beta(y)d(z) = 0 \ \forall \ y, r, z \in R$$

Since β is an automorphism of R and d is a skew derivation of R and $x. y, r, z \in R$

This implies that,
$$d(r)R \ d(z) \ \ \forall \ r, z \in R$$

By primeness of R, this implies that

$d(r) = 0$ or $d(z) = 0$. Hence, we obtained the require result. Using similar approach we can prove that the same result holds for

$$d(x)d(y) - xy = 0 \forall \ x, y, r, z \in R.$$

Theorem 4
Let R be prime ring and d be a skew- derivative associated with an automorphism $\beta: R \to R$
If $d(y)d(x) \pm yx = 0$ for all $x, y \in R$, then $d = 0$.

Proof: first we consider the case

$$d(y)d(x) + yx = 0 \tag{19}$$

for all $x, y, \in R$. Substituting xz instead of x in equation (19), we obtain
$$d(y)d(xz) + yxz = 0$$

By definition of skew derivation, we get

$$d(y)\big(d(x)z + \beta(x)d(z)\big) + yxz = 0 \ \forall \ x, y, z \in R$$
$$d(y)d(x)z + d(y)\beta(x)d(z) + yxz = 0$$

But $d(y)d(x) = -xyz$
$$-yxz + d(y)\beta(x)d(z) + yxz = 0$$
$$d(y)\big(\beta(x)d(z)\big) + yxz - yxz = 0$$
$$d(y)\big(\beta(x)d(z)\big) = 0 \tag{20}$$

Replacing yr instead y in equation (20), we obtain
$$d(yr)\big(\beta(x)d(z)\big) = 0$$

By definition of skew derivation, we have

$$(d(y)r + \beta(y)d(r))\big(\beta(x)d(z)\big) = 0$$
$$d(y)r\beta(x)d(z) +$$
$$\beta(y)d(r)\beta(x)d(z) = 0 \tag{21}$$

Replacing $r\beta(x)$ instead of $\beta(x)$ in equation (20), we get

$$d(y)r\beta(x)d(z) = 0 \ \forall \ x. y, r, z \in R \tag{22}$$

Subtracting equation (22) from equation (21), we obtain

$$\beta(y)d(r)\beta(x)d(z) = 0 \ \forall \ x. y, r, z \in R \tag{23}$$

Replacing $d(r)$ instead of $\beta(y)d(r)$ in equation (23), we get
$$d(r)\beta(x)d(z) = 0 \ \forall \ y, r, z \in R$$

Since β is an automorphism of R and d is a skew derivation of R and $x. y, r, z \in R$

This implies that,
$$d(r)R \ d(z) \ \ \forall \ r, z \in R$$

By primeness of R, this implies
$d(r) = 0$ or $d(z) = 0$. Hence, we obtained the require result. Using similar approach we can prove that the same result holds for

$$d(y)d(x) - yx = 0 \forall \ x, y, r, z \in R.$$

Conclusion
In this paper, commutativity of prime rings with multiplicative (generalized) reverse-derivations and reverse skew-derivation is established. We proved that prime rings that admit a nonzero multiplicative reverse-derivation satisfying certain algebraic (or differential) identities are commutative rings.

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