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Full Length Research Article

A STUDY ON THE EFFICIENCY OF NEW BETA POLYNOMIAL KERNELS

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ABSTRACT
Kernel density estimation is a widely used nonparametric method for estimating the probability density functions of observed data. The efficiency of the kernel method is significantly influenced by the choice of the kernel function and other statistical properties such as its roughness and variance. This study investigated the new beta polynomial family's efficiency and compared the efficiency values with the classic beta kernel family. The roughness and variance of the new core functions were determined to calculate the efficiency values. The numerical values of the efficiency of the classic beta family and the new family were determined and compared for univariate and bivariate kernel functions. The results of the study showed that the new beta family has higher efficiency values compared to the classic beta family. The higher efficiency of the proposed beta family is due to their coefficients being larger than the classic kernel functions.

Keywords: Bandwidth, Beta Kernel, Density Estimation, Efficiency.

INTRODUCTION.

Estimating the underlying probability density function of a data set is a fundamental task in statistics. Accurate density estimation provides important insight into the structure and distribution of data, enabling informed decision making in diverse fields such as economics, biology, and engineering. Among the numerous techniques developed for density estimation, the kernel density estimation method has emerged as a versatile and powerful nonparametric method for approximating probability density function based on observed data (Yuan *et al*., 2020; Choi *et al*., 2022; Siloko and Uddin, 2023; Tseng and Yang, 2023).

Kernel density estimation offers several advantages over traditional parametric approaches. Unlike the parametric methods, which assume a specific functional form for the underlying distribution, the kernel density approach makes no assumptions about the data structure. This flexibility allows the kernel density method to effectively model complex and multimodal distributions by adapting to the intrinsic properties of the data without being limited by predefined distribution shapes. Consequently, the kernel method has been widely used in various applications including data visualization, signal processing, anomaly detection and pattern recognition (Siloko and Siloko, 2023; Neto *et al*., 2024; Siloko *et al*., 2024).

Despite its widespread use and effectiveness, the kernel method is not without challenges. One of the main difficulties lies in the selection of the appropriate bandwidth and kernel functions, which significantly influence the bias and variance of the estimator. Inadequate choice of these parameters can lead to either excessive smoothing, obscuring important features of the data, or insufficient smoothing, resulting in noisy and unreliable estimates (Dhaker *et al*., 2018; Tsuruta and Sagae, 2020). Furthermore, kernel density estimation performance may degrade in high-dimensional environments due to the curse of dimensionality associated with nonparametric estimation, necessitating the development of advanced techniques and adaptations to mitigate these problems.

In recent years, significant efforts have been made to improve the performance and applicability of kernel estimation, such as adaptive bandwidth selection methods, the introduction of novel kernel features, and the integration of kernel methods into other statistical and machine learning frameworks (Fuentes-Santos *et al*., 2023; Govorov *et al*., 2023). In addition, computational improvements have been proposed to address the scalability issues associated with large datasets by using techniques such as fast Fourier transforms and efficient data structures to accelerate density estimation processes (Zámečník *et al*., 2023). The aim of this study is to investigate the efficiency of newly developed beta polynomial kernels in kernel density estimation. The study seeks to evaluate how the new beta polynomial kernels improve upon the traditional kernels in terms of efficiency numerically.

MATERIALS AND METHODS

The Kernel Density Estimator

The kernel density estimator is a nonparametric statistical method for modelling many real-world situations. Since its introduction by Rosenblatt (1956) and Parzen (1962), the estimator has gained popularity in many areas of study, especially for data exploratory analysis and visualization purposes (Sheather, 2004; Siloko *et al.,* 2021; Somé and Kokonendji, 2022). The univariate form of the kernel estimator is

$$\hat{f}(\mathrm{x}) = \frac{1}{nh_{\mathrm{x}}}\sum_{i=1}^{n} K\left(\frac{\mathrm{x}-\mathrm{X}_i}{h_{\mathrm{x}}}\right), \quad (1)$$

where $K(\cdot)$ is a kernel function and h_{x} is bandwidth which regulates the smoothness of the estimate, n is sample size, x is the data range and X_i are the observations (Silverman, 2018). The kernel estimator satisfies the condition

$$\int K(\mathrm{x})d\mathrm{x} = 1. \quad (2)$$

Generally, the kernel function $K(\mathrm{x}) \geq 0$ is a symmetric probability density function. Every kernel function is a probability density function whose integral is one and having a mean of zero with variance greater than zero. The two essential factors in kernel density estimation are choice of bandwidth and kernel function. On the choice of bandwidth, several authors have introduced novel selectors with little researches on kernel function (Siloko *et al.,*

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2018; Wang *et al.,* 2020; Xie *et al.,* 2023). The bandwidths determine the performance of kernel estimator and a popular assessment criterion of the kernel estimator is the mean integrated squared error whose univariate form is

$$MISE\left(\hat{f}(\mathrm{x})\right) = \frac{R(K)}{nh_{\mathrm{x}}} + \frac{1}{4}\mu_2(K)^2 h_{\mathrm{x}}^4 R(f'') + O\left(\frac{1}{nh_{\mathrm{x}}} + h_{\mathrm{x}}^4\right), \quad (3)$$

where $R(K)$ is roughness of kernel, $\mu_2(K)^2$ is variance of kernel while $R(f'') = \int f''(\mathrm{x})^2 dx$ is roughness of the unknown function (Wand and Jones, 1995; Scott, 2015; Silverman, 2018). The approximate form known as the asymptotic mean integrated squared error is

$$AMISE\left(\hat{f}(\mathrm{x})\right) = \frac{R(K)}{nh_{\mathrm{x}}} + \frac{1}{4}\mu_2(K)^2 h_{\mathrm{x}}^4 R(f''). \quad (4)$$

The bandwidth that minimizes the AMISE in Equation (4) called the optimal bandwidth is

$$h_{\mathrm{x-AMISE}} = \left[\frac{R(K)}{\mu_2(K)^2 R(f'')}\right]^{1/5} \times n^{-1/5}. \quad (5)$$

Furthermore, the two-dimensional kernel estimator is given as

$$\hat{f}(\mathrm{x}\,,\mathrm{y}) = \frac{1}{nh_{\mathrm{x}}h_{\mathrm{y}}}\sum_{i=1}^{n} K\left(\frac{\mathrm{x}-\mathrm{X}_i}{h_{\mathrm{x}}}, \frac{\mathrm{y}-\mathrm{Y}_i}{h_{\mathrm{y}}}\right) = \frac{1}{nh_{\mathrm{x}}h_{\mathrm{y}}}\sum_{i=1}^{n} K\left(\frac{\mathrm{x}-\mathrm{X}_i}{h_{\mathrm{x}}}\right) K\left(\frac{\mathrm{y}-\mathrm{Y}_i}{h_{\mathrm{y}}}\right), \quad (6)$$

where, $h_{\mathrm{x}} > 0$ and $h_{\mathrm{y}} > 0$ are the bandwidths of X and Y with x and y representing the ranges of the observations in the various axes (Siloko *et al.*, 2023; Gündüz and Karakoç, 2023). With the bivariate product kernel estimator, different smoothing parameters can be used for the different axes, which is particularly advantageous if the scales of the variables in the respective axes vary. When presented as surface or contour plots, the bivariate kernel density estimates are easy to use and understand. An effective tool for exploratory data analysis and visualization is the bivariate kernel estimator (Silverman, 2018). The AMISE of the two-dimensional kernel estimator employing the product kernel is

$$AMISE\left(\hat{f}(\mathrm{x},\mathrm{y})\right) = \frac{R(K)}{nh_{\mathrm{y}}h_{\mathrm{z}}} + \frac{h_{\mathrm{y}}^4}{4}\mu_2(K)^2 \int\int \left(\frac{\partial^2 f}{\partial \mathrm{y}^2}\right)^2 dydz + \frac{h_{\mathrm{z}}^4}{4}\mu_2(K)^2 \int\int \left(\frac{\partial^2 f}{\partial \mathrm{z}^2}\right)^2 dydz \quad (7)$$

Similarly, the bandwidths of the two-dimensional product kernel estimator are given by

$$h_{\mathrm{x}-AMISE} = \left\{\frac{dR(K)^d}{\mu_2(K)^2 \left(\frac{d(d+2)}{4(2\sqrt{\pi})^d}\right) \sigma_{\mathrm{x}}^{-(d+4)}}\right\}^{1/6} \times n^{-1/6} \quad (8)$$

$$h_{y-AMISE} = \left\{\frac{dR(K)^d}{\mu_2(K)^2 \left(\frac{d(d+2)}{4(2\sqrt{\pi})^d}\right) \sigma_{y}^{-(d+4)}}\right\}^{1/6} \times n^{-1/6}, \quad (9)$$

where σ_{x} and σ_{y} are the standard deviations of variables X and Y respectively while d is the dimension of the kernel function (Scott 2015; Siloko *et al.*, 2022).

The Beta Polynomial Kernel Function

The beta polynomial kernel family is one of the prominent families of kernel estimators and its general form is

$$K_p(t) = \frac{(2p+1)!}{2^{2p+1}(p!)^2}(1 - t^2)^p, \quad (10)$$

where $p = 0, 1, 2, \ldots$ determines the resulting kernel such that t is within the interval $-1 \leq t \leq 1$. The fascinating mathematical attributes of the beta kernel contributed to their popularity in density estimation (Duong, 2015). The resulting kernel members of this family are determined by the value of p such that when $p = 0$, we have the simplest kernel called the Uniform kernel and when $p = 1$, the resulting kernel is the Epanechnikov function known as the optimal kernel with reference to the AMISE. Again, when $p = 2, 3$ and 4, the corresponding kernel functions are Biweight, Triweight and Quadriweight kernels. The popular Gaussian kernel which is of wide applicability in statistical estimation did not belong to this family, however; it is obtained when p tends to infinity (Marron and Nolan, 1988). The mathematical expression of the Epanechnikov, Biweight, Triweight and Quadriweight kernels are

$$\begin{cases} K_1(t) = \frac{3}{4}(1-t^2). \\ K_2(t) = \frac{15}{16}(1-t^2)^2. \\ K_3(t) = \frac{35}{32}(1-t^2)^3. \\ K_4(t) = \frac{315}{256}(1-t^2)^4. \end{cases} \quad (11)$$

The Epanechnikov kernel, also known as quadratic kernel, Biweight kernel, also called quartic kernel, Triweight and Quadriweight kernels, except for the Uniform kernel function, are widely used in statistical estimation. The Epanechnikov kernel is usually used in calculating the efficiency of the beta polynomial kernels due to its optimality property with respect to AMISE. A new family of beta polynomial kernel functions was developed by Siloko *et al.* (2020) using exponential progression in their derivation. The new beta kernel family, whose results compete well with the classic families, is given by

$$K_p(t) = \left(\frac{3}{4}(1-t^2)\right)\left(\frac{3+2p}{2+2p}(1-t^2)\right)^{p-1}, \quad (12)$$

where $p = 0, 1, 2, 3, \ldots$ and t is the value where the kernel is evaluated, typically in the interval [−1, 1]. Again, as in the classical beta kernels, when $p = 0$, the resulting kernel from Equation (12) is the Uniform kernel while $p = 1$, resulted in the optimum kernel which is the Epanechnikov kernel. Nevertheless, when $p = 2, 3, 4, \ldots$, the new kernels of Biweight, Triweight, and Quadriweight kernel functions are as follows

$$K_2(t) = \frac{7}{8}(1-t^2)^2. \quad (13)$$

$$K_3(t) = \frac{243}{256}(1-t^2)^3. \quad (14)$$

$$K_4(t) = \frac{3993}{4000}(1-t^2)^4. \quad (15)$$

Although the powers of the two families are the same, the new kernel family and the classic beta kernel functions differ in the value

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of the normalization constants. The value of the AMISE performance measure changed in parallel with the changes in the normalization constant. The choice of a kernel function is based on its performance, and a method or kernel function is considered superior if it produces a lower AMISE value (Jarnicka, 2009). Furthermore, Equation (12) has been generalized to the multivariate case using the product kernel approach. See (Siloko *et al*., 2020; Siloko *et al*., 2023) for a detailed derivation of the new kernel family from the existing beta polynomial family.

The Efficiency of Kernel Function.
The efficiency of the symmetric kernel is evaluated by comparing with the Epanechnikov kernel function. The efficiency of any kernel function is derived from the relation given as

$$Eff(K) = \left(\frac{C(K_e)}{C(K)}\right)^{1/4} = \left(\frac{R(K_e)^4 \mu_2(K_e)^2}{R(K)^4 \mu_2(K)^2}\right)^{1/4}, \qquad (16)$$

where $C(K) = R(K)^4 \mu_2(K)^2$ is a constant of any given kernel and $C(K_e) = R(K_e)^4 \mu_2(K_e)^2$ is the constant of Epanechnikov kernel (Qahtan, 2017; Siloko *et al.,* 2019). The efficiency of the multi-dimensional kernel with the product strategy is

$$Eff(K^p) = \left\{\left(\frac{C(K_e^p)}{C(K^p)}\right)^{1/(d+4)}\right\}^{(d+4)/4}$$
$$= \left(\frac{R(K_e^p)^4 \mu_2(K_e)^2}{R(K^p)^4 \mu_2(K)^2}\right)^{1/4}, \qquad (17)$$

where d is the kernel dimension, while $C(K_e^p)$ is the higher dimensional product form of the Epanechnikov kernel constant and $C(K^p)$ is the higher dimensional form of every other kernel function in the family (Silverman, 2018). Calculating the efficiency of any kernel function requires two important statistics, namely the roughness of the kernel function and its variance, which can be seen in Equation (16). The roughness of the kernel function is given by

$$R(K) = \int K(t)^2 dt. \qquad (18)$$

Similarly, the second moment of any kernel function also known as the variance is of the form

$$\mu_2(K) = \int t^2 K(t) dt. \qquad (19)$$

The Epanechnikov kernel is considered to be the optimal kernel in terms of the asymptotic mean integrated squared error because it gives the smallest AMISE value when applied to the classical second-order kernel.

RESULTS AND DISCUSSION
Mathematica version 12 software is used for graphical analysis and calculation of the efficiency of kernel functions. The statistical properties of p are examined, for which $p = 1, 2, 3, 4$ and which represent the Epanechnikov, Biweight, Triweight and Quadriweight kernels. These kernel functions are of great use and form the basis for discussing the beta kernel family, especially the Epanechnikov kernel, in calculating the efficiency of other kernel functions in this family. The performance of the kernel function is determined by its efficiency. In contrast to univariate kernel functions, the efficiency values of bivariate beta polynomial kernel functions were determined using the product approach.
The efficiency values of the proposed and classical beta polynomial kernel functions were numerically calculated. Each kernel function in the beta polynomial family was evaluated for efficiency by comparing it with the Epanechnikov kernel. The results in Table 1 and Table 2 are the efficiency values of the classical and the introduced kernel functions of the beta polynomial kernels for the univariate and bivariate cases. As can be seen in Table 1 and Table 2, the efficiency values of the proposed kernels are larger than those of the classic kernels, which shows that the proposed kernel outperforms the traditional kernels of this family. The better performance in terms of efficiency is due to the size of the normalization constant. The normalization constant is the coefficient of the beta kernel functions. Regarding AMISE as a performance measure, the proposed kernels have demonstrated superiority over their classical counterparts (see Siloko *et al.,* 2020; Siloko *et al.,* 2023).
The calculated statistical properties of the two-dimensional product kernels for the proposed and classical beta polynomial kernels are shown in Table 2. Some kernel functions in the beta family have efficiencies less than one, while the Epanechnikov kernel has an efficiency of one. From Tables 1 and 2, it is clear that as kernel power increases, the efficiency of the proposed kernels increases, while there is loss of efficiency in the traditional kernel functions. The larger normalization constant of the proposed kernels explains their superior efficiency.

Table 1: Univariate Efficiencies of Classical Kernel Functions and Proposed Kernel Functions

Kernel Functions	Classical Kernel Functions			Proposed Kernel Functions		
$K(t)$	$R(K)$	$\mu_2(K)$	$Eff(K)$	$R(K)$	$\mu_2(K)$	$Eff(K)$
$K_1(t)$	$\frac{3}{5}$	$\frac{1}{5}$	1.000	$\frac{3}{5}$	$\frac{1}{5}$	1.000
$K_2(t)$	$\frac{5}{7}$	$\frac{1}{7}$	0.994	$\frac{28}{45}$	$\frac{2}{15}$	1.181
$K_3(t)$	$\frac{350}{429}$	$\frac{1}{9}$	0.987	$\frac{19683}{32032}$	$\frac{27}{280}$	1.406
$K_4(t)$	$\frac{2205}{2431}$	$\frac{1}{11}$	0.981	$\frac{596982}{1000000}$	$\frac{968}{13125}$	1.655

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Table 2: Bivariate Efficiencies of Classical Kernel Functions and Proposed Kernel Functions

Kernel Functions	Classical Kernel Functions			Proposed Kernel Functions		
$K(t)$	$R(K)$	$\mu_2(K)$	$Eff(K)$	$R(K)$	$\mu_2(K)$	$Eff(K)$
$K_1(t)$	$\frac{9}{25}$	$\frac{1}{25}$	1.000	$\frac{9}{25}$	$\frac{1}{25}$	1.000
$K_2(t)$	$\frac{127551}{250000}$	$\frac{2551}{125000}$	0.988	$\frac{38716}{100000}$	$\frac{4}{225}$	1.394
$K_3(t)$	$\frac{665613}{1000000}$	$\frac{6173}{500000}$	0.974	$\frac{377585}{1000000}$	$\frac{93}{1000}$	1.977
$K_4(t)$	$\frac{205677}{250000}$	$\frac{1033}{125000}$	0.963	$\frac{356386}{1000000}$	$\frac{5439}{1000000}$	2.739

The univariate plots of the classic and proposed beta kernels are shown in Figure 1 and Figure 2, respectively. The plots are evaluated within the interval $-1 \leq t \leq 1$, that is, $t \in [-1, 1]$. All and the statistical properties such as the roughness and variance of the kernel for calculating bandwidths and efficiency of the kernels are numerically integrated within the interval. As the power beta polynomial kernels are normally evaluated within this interval

of the kernel functions increases, the narrower the peak of the graphs as seen in Figure 1 and Figure 2.

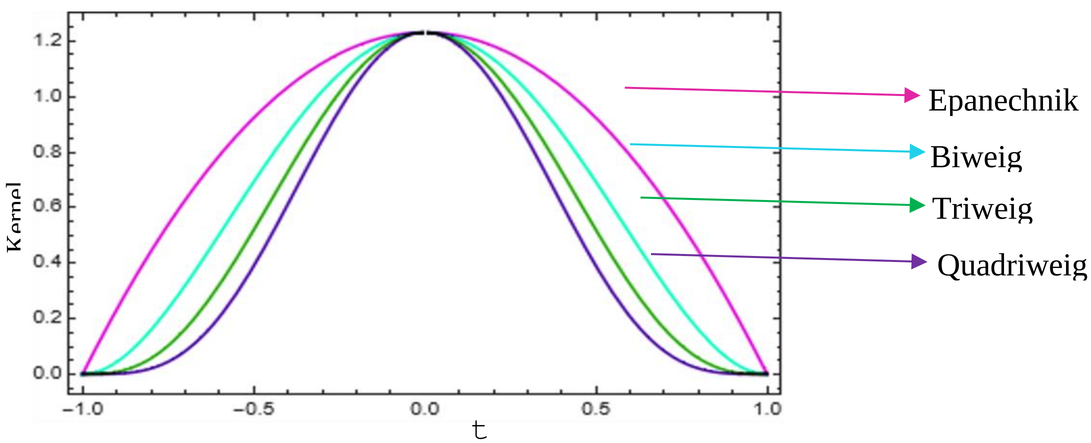

Figure 1: Graphs of Classical Beta Polynomial Kernel

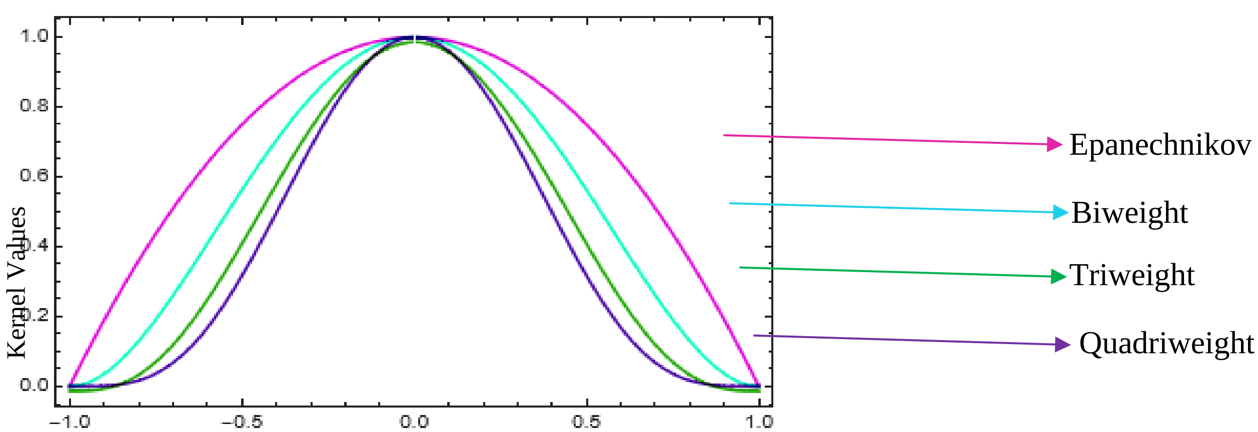

Figure 2: Graphs of Proposed Beta Polynomial Kernel Functions

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The bivariate plots in Figures (3–6) show the conventional and novel product core features. As can be seen from the estimates of the traditional and recently introduced families, the loops of the graphs for the Triweight and Quadriweight kernels moved closer to the centre than the loops for the Epanechnikov and Biweight kernels. The degree of differentiability causes the loop to move to the centre of the graph in situations where p is larger. Higher p-kernel functions, due to their higher derivative and large normalization constants, tend to produce graphs with loops closer to the centre and consequently better kernel estimates in terms of smoothness.

Despite the similarity presented in the univariate and bivariate plots of the traditional and proposed kernels, the empirical results showed that the proposed kernels outperformed the classical version using AMISE, and this is due to the variation of their normalization constants (see Siloko *et al.*, 2020; Siloko *et al.*, 2023). The newly developed beta polynomial kernels are valuable addition to the family of kernel methods, offering better performance in relation to the AMISE and maintaining computational feasibility in terms of efficiency values (Jarnicka, 2009). The introduced beta polynomial kernels have strong potential for adoption in practical applications with bounded data distributions.

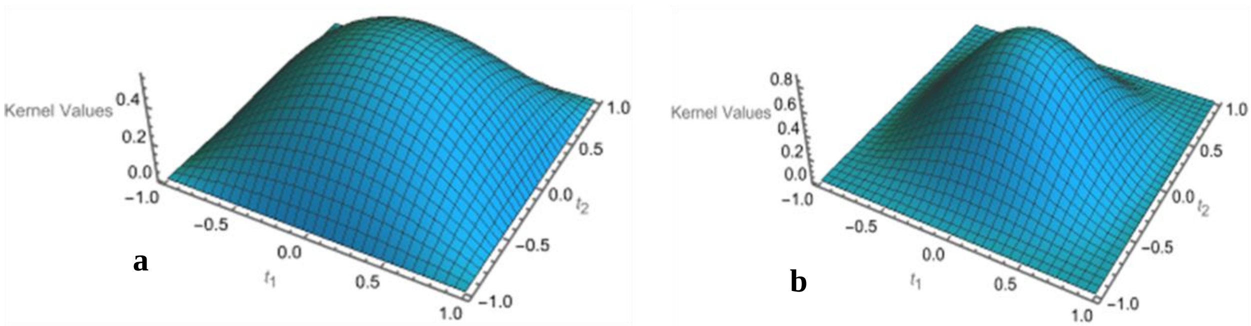

Figure 3: Bivariate Estimate of the Classical Epanechnikov Function **(a)** and Biweight Function **(b)**

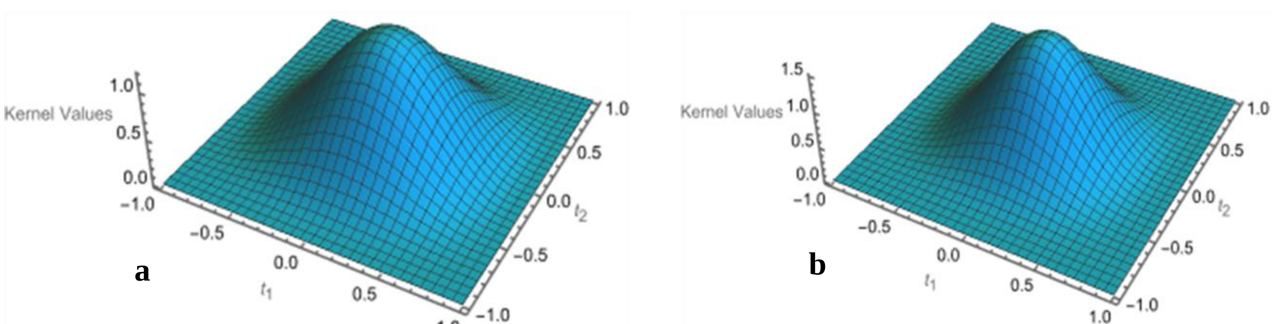

Figure 4: Bivariate Estimate of the Classical Triweight Function **(a)** and Quadriweight Function **(b)**

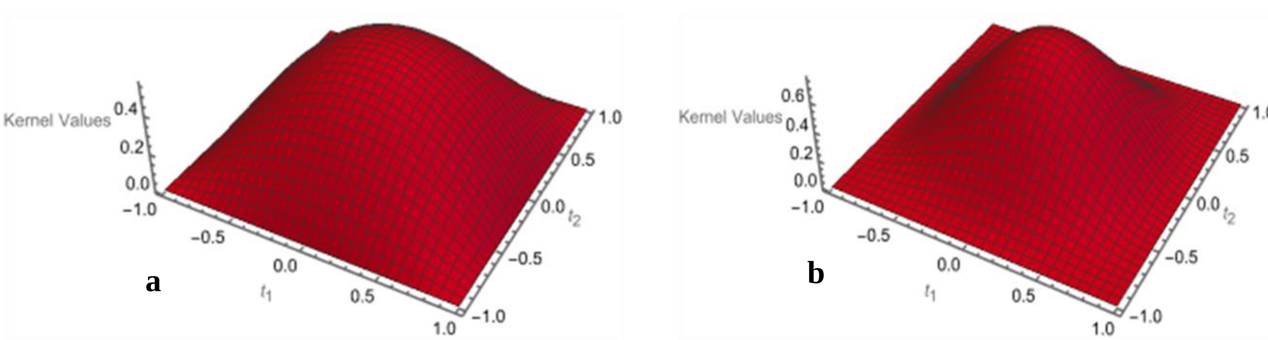

Figure 5: Bivariate Estimate of the Proposed Epanechnikov Function **(a)** and Biweight Function **(b)**

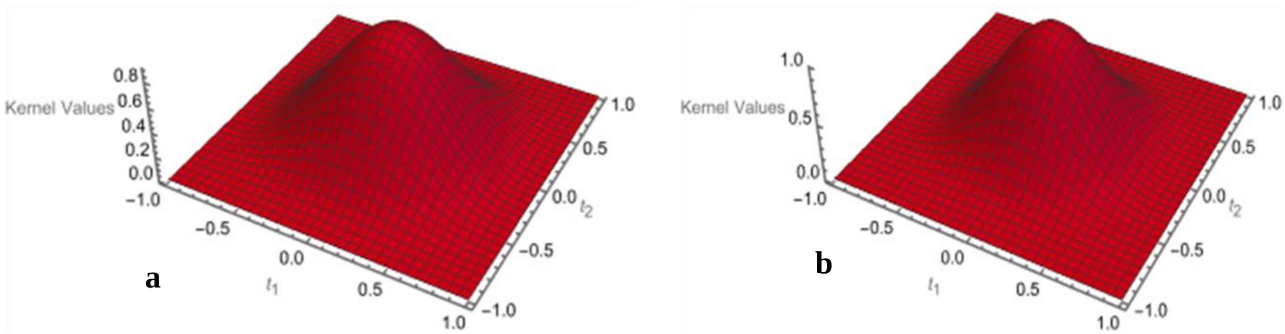

Figure 6: Bivariate Estimate of the Proposed Triweight Function **(a)** and Quadriweight Function **(b)**

Conclusion

This paper has examined the efficiency of a newly introduced kernel family which has the potential to be widely used in a variety of statistical and data analysis tasks. The newly developed beta polynomial kernels have shown to be highly effective, particularly in their efficiency values. Due to the polynomial nature of the beta kernels, the newly introduced kernels also provided a high-level smoothness, making them more versatile across different datasets. The choice of a kernel function should be based on the degree of differentiability, since kernels with higher powers tend to be smoother and have more derivatives, as opposed to those with fewer derivatives, which tend to be noisy. The results of the study suggest that the new beta family outperforms the traditional beta family in terms of efficiency. The higher efficiency values of the proposed beta family are explained by the fact that the coefficient of the proposed kernels is larger than the traditional kernels.

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Conflict of Interest

The authors declared that they have no conflict of interest.

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