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Full Length Research Article

ON THE FIXED POINTS THEORY OF STRONG PARTIAL B- METRIC SPACES

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ABSTRACT

The paper resorted to some fixed point results for Kannan type contraction in Strong Partial b-Metric Spaces. It is a generalization of metric space and strong b- metric spaces**.** As proves of unique fixed point theorems for a Kannan mapping in a complete metric spaces is presented. We provided some examples to illustrate our results and demonstrate how valid the result is, with suitable examples.

Keywords: Fixed point, Kannan contraction, B-Metric spaces, completeness

INTRODUCTION

The Banach contraction principle Banach, (1922) has gained remarkable attention from researchers in mathematics. Its contractive condition on the mapping presents a good analytical framework. As a contractive principle requires a complete metric space Petrov, (2023), Kumari *et al.*, (2023), Moshokoa & Ncogwane, (2020), Savaliya *et al.,* (2024) as the principle to the study of the existence and uniqueness of solutions. Obtaining the extension of the contractive condition through expansion of the condition of the mapping Grnicki, (2018). A metric spaces is complete if and only if every Kannan mapping has a fixed point Mathews, (1994). Completeness in strong b- metric spaces as in Dehici *et al.*, (2019), Moshokoa & Ncogwane, (2020), Doan, (2021), Wang *et al.,* (2024) prove the uniqueness of the fixed point. And extensions of Kannan fixed point theory and applications can be seen in Berinde & Pacurar (2019), Kannan, (1968), Petrov & Bisht, (2023), Petrov, (2023), Grnicki, (2018), Doan, (2021), Wang *et al.*, (2024), Pant, (2024). There are several generalizations of contractive mapping principle.

MATERIALS AND METHODS

Definition 1: Kirk & Shahzad, (2014). Let a map d : E × E → R be a strong b- metric on a non-empty set E if for u, v, c ∈ E and for any $\mu \geq 1$ the following conditions are met,

i. u = v iff d(u, v) = 0
ii. d(u, v) = d(v, u)
iii. d(u, v) ≤ d(u, c) + μ d(c, v)
The triple (E, d, μ) is called a strong b-metric space.

Definition 2. Mathews, (1994) A function d : E × E → R is a partial metric on a set E, such that for all u, v, c ∈ E, the following conditions are met.

i. u= v iff d(u, u) = d(v, v) = d(u, v);
ii. d(u, u) ≤ d(u, v) ;
iii. d(u, v) = d(v, u).
iv. d(u, v) ≤ d(u, c) + d(c, v) – d(c, c).
hence, (E, d) is called a partial metric space.

Definition 3. Moshokoa & Ncogwane, (2020). Let a map d : E × E → R is a strong partial metric on non empty set E, given that for all u, v, c ∈ E and $\mu \geq 1$ the following conditions are satisfied;

i. u= v iff d(u, u) = d(v, v) = d(u, v);
ii. d(u, u) ≤ d(u, v) ;
iii. d(u, v) = d(v, u)
iv. d(u, v) ≤ d(u, c) + μ d(c, v) – d(c, c).
hence, (E, d, μ) is called a strong partial b- metric space.

Definition 4. Let (X, d) be a metric space. A sequence (x_n) in X converges to the limit u as n→ ∞, where

x_n →u or $\lim_{n\to\infty} x_n = u$

Given that for every ε >0, there exist N∈ ℕ such that $|x_n - u| < \varepsilon$ ∀ n≥N.

Definition 5. Given a metric space (X, d). A sequence (x_n) in X is said to be Cauchy sequence if for every ε >0, there exist for m, n ≥ N as N∈ ℕ such that $|x_m - x_n| < \varepsilon$.

Definition 6. A function g : ℝ → ℝ is continuous at some point u∈ ℝ if

$$\lim_{x\to u} g(x) = g(u)$$

Definition 7. A function g : ℝ → ℝ has the limit u as x→a, we write g(x)→L or $\lim_{x\to a} g(x)$ =L

If for every ε >0, there exist δ >0 such that $|g(x) - L| < \varepsilon$ and for $|x - a| < \delta$.

LEMMA 1. Dehici *et al.*, (2019). Let (X, d) be a metric space. Assume that G : X → X be a self-mapping on X satisfying that

d(Gx, Gy) ≤ α d(x, y) for all x, y ∈X. (1)

where α∈ [0, $\frac{1}{3}$[. Then, G is a Kannan mapping with a constant of contraction equal to $\frac{\alpha}{1-\alpha}$.

Proof

Let x_0 ∈X be any arbitrary point and $\{x_n\}$ be a sequence in X, such that

x_{n+1} = Gx_n ∀ n≥0

Given $x_{n+1} \neq x_n$ ∀ n≥0.

Let define F_n = d(x_{n+1}, x_n), ∀ n≥0

And by using the inequality (1), we have

$$F_{n+1} = d(x_{n+2}, x_{n+1}) = d(Gx_{n+1}, Gx_n) \leq \alpha\, d(x_{n+1}, x_n)$$
$$\leq \alpha\{ d(x_{n+2}, x_{n+1}) + d(x_n, x_{n+1})\}$$

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$$= \alpha\{F_{n+1} + F_n\}$$
$$1 - \alpha(F_{n+1}) \leq \alpha\{F_n\}$$
$$F_{n+1} \leq \frac{\alpha}{1-\alpha}\{F_n\}$$

Since, $[0, \frac{1}{3}[$, then $\frac{\alpha}{1-\alpha} \in [0, \frac{1}{2}[$. And as result, G is a Kannan mapping.

LEMMA 2. Dehici *et al.*, (2019). Let (X, d) be a metric space. Assume that G : X → X be a self-mapping on X satisfying that

d(Gx, Gy) ≤ α d(x, y) for all x, y ∈X (2)

where α∈ $[0, \frac{1}{3}[$. Then, G is a Kannan mapping with a constant of contraction equal to $\frac{\alpha}{1-\alpha}$.

Proof

Let x_0 ∈X be any arbitrary point and $\{x_n\}$ be a sequence in X, such that

$x_{n+1} = Gx_n$ ∀ n≥0

Given $x_{n+1} \neq x_n$ ∀ n≥0.

Let define $F_n = d(x_{n+1}, x_n)$, ∀ n≥0

And by using the inequality (2), we have

$$F_{n+1} = d(x_{n+2}, x_{n+1}) = d(Gx_{n+1}, Gx_n) \leq \alpha\, d(x_{n+1}, x_n)$$
$$\leq \alpha\{d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + d(x_n, x_{n+1})\}$$
$$= \alpha\{ F_n + F_{n+1} + F_n\}$$
$$1 - \alpha(F_{n+1}) \leq \alpha\{2F_n\}$$
$$F_{n+1} \leq \frac{\alpha}{1-\alpha}\{2F_n\}$$

Since, $[0, \frac{1}{3}[$, then $\frac{\alpha}{1-\alpha} \in [0, \frac{1}{2}[$. And as result, G is a Kannan mapping.

RESULTS AND DISCUSSION

Theorem 1. Dehici *et al.*, (2019). Let (X, d) be a complete metric space. G : X → X is contraction mapping if

d(Gx, Gy) ≤ α d(x, y)

For all x, y∈X, as α ∈(0, 1). Then G has a unique fixed point u∈X.

Theorem 2. Dehici *et al.*, (2019). Let (X, d) be a complete metric space and G : X → X be a selfmapping on X. Where there exists $\alpha \in [0, \frac{1}{2})$ such that

d(Gx, Gy) ≤ α [d(x, Gx) + d(y, Gy)] (1)

for all x, y∈X. Then G has a unique fixed point u ∈X.

Proof

Let x_0 ∈X be any arbitrary point and $\{x_n\}$ be a sequence in X, for all n≥0

As $x_{n+1} \neq x_n$ ∀ n≥0.

It follows from definition (1) that

$$d(x_{n+2}, x_{n+1})$$
$$\leq \alpha\, d(x_{n+1}, x_n)$$
$$\leq \alpha\{(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\}$$
$$d(x_{n+2}, x_{n+1}) \leq \frac{\alpha}{1-\alpha}\{d(x_{n+1}, x_n)\}$$

since condition (1) is satisfied and it is obvious that

$d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)$ ∀ n≥0

hence, $d(x_{n+1}, x_n)$ is monotonically decreasing and bounded below sequence. If there exist β ≥0 such that we have the

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \beta$$

Now, let assume β >0. Then, from the inequality (1), we get

$d(x_{n+2}, x_{n+1}) \leq \alpha_n \{ d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\}$

given that

$$\frac{d(x_{n+2},x_{n+1})}{d(x_{n+2},x_{n+1}) + d(x_{n+1},x_n)} \leq \alpha_n, \ \forall\ n\geq 0$$

as expression

$$\frac{d(x_{n+2},x_{n+1})}{d(x_{n+2},x_{n+1}) + d(x_{n+1},x_n)} \leq \lim_{n\to\infty} \alpha_n \leq \frac{d(x_{n+2},x_{n+1})}{d(x_{n+2},x_{n+1}) + d(x_{n+1},x_n)},$$

and this is a contradiction. Hence $\lim_{n\to\infty} \alpha_n = \beta = 0$.

And given that $\mu \in [0, \frac{1}{2})$ such that

$$d(x_n, x_{n+1}) \leq \mu\, d(x_{n-1}, x_n) \leq \ldots \leq \mu^n d(x_0, x_1) \quad \ldots (2)$$

given $G_n = d(x_n, x_{n+1})$ and $G_{n-1} = d(x_{n-1}, x_n)$, and reading from (2), we have

$$G_n \leq \mu G_{n-1} \leq \mu^2 G_{n-2} \leq \ldots \leq \mu^n G_0$$

We now demonstrate that $\{x_n\}$ is a Cauchy sequence in X. We let m>n and by definition (2) and (1), we get

$$d(x_n, x_m) \leq \{ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_m) \}$$
$$-\{ d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m-1}) \}$$
$$= \{ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_m) \}$$
$$\leq \mu^n d(x_0, x_1) + \mu^{n+1} d(x_0, x_1) + \ldots + \mu^{n+m-1} d(x_0, x_1)$$
$$= \mu^n [d(x_0, x_1) + \mu d(x_0, x_1) + \ldots + \mu^{m-1} d(x_0, x_1)]$$
$$= \mu^n [1 + \mu + \ldots \mu^{m-1}]\, G_0$$

Applying n, m → ∞ as $d(x_n, x_m)$ →0, for $\mu \in [0, \frac{1}{2})$, hence $\{x_n\}$ is a Cauchy sequence in X. In addition, since (X, d) is complete,

We now by (iv) of Definition 3,

$$d(x_n, x_m) \leq \alpha\{ d(x_{n-1}, x_m) + d(x_m, x_n)\} + \mu\, d(x_{n-1}, x_n) - d(x_{n-1}, x_{n-1})$$
$$d(x_n, x_m)(1-\alpha) \leq \alpha\{ d(x_{n-1}, x_m)\} + \mu\, d(x_{n-1}, x_n)$$
$$\leq \frac{\alpha}{1-\alpha} \{ d(x_{n-1}, x_m) \} + \frac{\mu}{1-\alpha} d(x_{n-1}, x_n) \quad (3)$$

And as n, m → ∞, the right hand side of (3) moves to zero.

so there exist u∈X such that x_n → u, as n→ ∞, x∈X, we have

$$d(Gu, u) = \lim_{n\to\infty} d(u, x_n) = \lim_{n,m\to\infty} d(x_n, x_m) = 0 \quad (4)$$

At this point, we observe that by (4), d(Gu, u) =0, we are required to demonstrate that u is a fixed point of G. By (ii) of definition (3), we have

d(Gu, Gu)≤ d(Gu, u)

and since d(Gu, u) = 0 means d(Gu, Gu) =0 as d(u, x) =0. Thus, we have

d(Gu, Gu)= d(Gu, u) = d(u, u)

so we have Gu = u by (i) of definition 3. Hence, u is a fixed point of G.

Uniqueness: Let v be another fixed point of G with u≠v, we have

d(u, v) = d(Gu, Gv)

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$\leq \alpha$\{d(u, v) +d(Gu, Gv)\}
And by the inequality
d(u, v) $\leq \mu$d(u, v)
Implies
d(u, v) = 0
as $\mu \in [0, \frac{1}{2})$
Which implies that u=v, thus the fixed point of G, is unique.

Example1. Let E = \{0, 1, 2\} and d: E×E → [0, ∞) be defined by
d(0, 0) = d(1, 1) = 0, d(2, 2) = $\frac{1}{3}$
•d(1, 0) = d(0, 1) = $\frac{2}{3}$
•d(1, 2) = d(2, 1) = 4
•d(2, 0) = d(0, 2) = 7
where we have d(u, u)<d(u, v), ∀ u, v ∈ E

1. d(0, 1) ≤ d(0, 2) + αd(2, 1) – d(2, 2), ∀ $\alpha \geq 1$
2. d(1, 0) ≤ d(1, 2) + αd(2, 0) – d(2, 2), ∀ $\alpha \geq 1$
3. d(1, 2) ≤ d(1, 0) + αd(0, 2) – d(0, 0), ∀ $\alpha \geq 1$
4. d(2, 1) ≤ d(2, 0) + αd(0, 1) – d(0, 0), ∀ $\alpha \geq 1$
5. d(2, 0) ≤ d(2, 1) + αd(1, 0) – d(1, 1), ∀ $\alpha \geq \frac{9}{2}$
6. d(0, 2) ≤ d(0, 1) + αd(1, 2) – d(1, 1), ∀ $\alpha \geq \frac{19}{3}$

The result indicate (E, α, d) is a Strong Partial b-Metric Space, where $\alpha = \frac{19}{3}$ but it is neither strong b metric nor metric space as d(2, 2) = $\frac{1}{3} \neq 0$.
So, the above cannot be applied to theorem 2, therefore let's T : E → E be a self map defined by T0 = 0, T1 =0, T2 =1 and μ ∈G defined by
$\mu(x) = \frac{1}{2}\sqrt[2]{2^{-\frac{x}{5}}}$ for x>0 and $\mu(0) \in [0, \frac{1}{2})$
then
• d(T0, T1) = d(0, 0) = 0 < 0.3180 = $\frac{1}{3}\sqrt[2]{2^{-\frac{2}{15}}}$ = μ(d(0, 1))\{d(0, T0) + d(1, T1)\}
• d(T1, T2) = d(0, 1) = $\frac{2}{3}$ < 1.7683 = $\frac{7}{3}\sqrt[2]{2^{-\frac{4}{5}}}$ = μ(d(1, 2))\{d(1, T1) + d(2, T2)\}
• d(T0, T2) = d(0, 1) = $\frac{2}{3}$ < 1.2311 = 2 $\sqrt[2]{2^{-\frac{7}{5}}}$ = μ(d(0, 2))\{d(0, T0) + d(2, T2)\}
therefore, we have G meeting all the conditions of theorem 2 and has a fixed point u =0.
Example 2. Given T : E → E and v, u ∈ [0, $\frac{1}{2}$[,we have

$$Tv = \begin{cases} \frac{v}{4} \ , if \ \ v \in [0, \ \frac{1}{2}[\\ \frac{1}{8} \ , if \ \ v = \ \frac{1}{2} \end{cases}$$

Let v, u ∈ [0, $\frac{1}{2}$[. Thus

$$|Tv - Tu| = \left|\frac{v}{4} - \frac{u}{4}\right| = \frac{1}{4}|v - u|$$

and

$$|v - Tv| = \left|\frac{v}{4} - v\right| = \frac{3}{4}v, \quad |u - Tu| = \frac{3}{4}u$$

which implies that

$$|Tv - Tu| = \frac{1}{4}|v - u| \leq \frac{30}{89}(|v - Tv| + |u - Tu|)$$

now, if v ∈ [0, $\frac{1}{2}$[and u = $\frac{1}{2}$, we get

$$|Tv - Tu| = \left|\frac{v}{4} - \frac{1}{8}\right|$$

$$|v - Tv| = \frac{3}{4}v, \quad |T1 - 1| = \frac{7}{8}$$

Consequently, we have v, u ∈ [0, $\frac{1}{2}$], and thus

$$|Tv - Tu| \leq \frac{v}{4} + \frac{1}{8} \leq \frac{30}{89}(|v - Tv| + |u - Tu|)$$

for
$\mu = \frac{30}{89} \in (0, \frac{1}{2})$

Theorem 4 Dehici *et al*., (2019). Let (X, d) be a complete metric space and G : X → X be a selfmapping on X. Where there exists $\alpha \in [0, \frac{1}{2})$ such that
d(Gx, Gy) $\leq \alpha$[d(x, Gx) + d(y, Gy) + d(x, y)] (5)
for all x, y∈X. Then G has a unique fixed point u ∈X.
Proof
Let x_0 ∈X be any arbitrary point and $\{x_n\}$ be a sequence in X, for all
n≥0
and as $x_{n+1} \neq x_n$ ∀ n≥0.
It follows from definition (1) that
d(x_{n+2}, x_{n+1})
$\leq \alpha$ d(x_{n+1}, x_n)
$\leq \alpha$\{d(x_n, x_{n+1}) + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\}
d(x_{n+2}, x_{n+1}) $\leq \frac{\alpha}{1-\alpha}$\{d($x_{n+1}, x_n$) + d($x_n, x_{n+1}$)\}
since (1) is satisfied and it is obvious that
d(x_{n+2}, x_{n+1})< d(x_{n+1}, x_n) ∀ n≥0
hence, d(x_{n+1}, x_n) is monotonically decreasing and bounded below sequence. If there exist $\beta \geq 0$ such that we have the
$\lim_{n\to\infty}$ d$(x_{n+1}, x_n) = \beta$
Now, let assume β >0. Then, from theorem (4),
d($x_{n+2}, x_{n+1} \leq \alpha_n$ \{ d(x_n, x_{n+1}) +d(x_{n+1}, x_{n+2})+d(x_{n+1}, x_n)\} (6)
given that

$$\frac{d(x_{n+2}, x_{n+1})}{\{ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_n)\}} \leq \alpha_n,$$

∀ n≥0
taking

$$\frac{d(x_{n+2}, x_{n+1})}{\{ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_n)\}} \leq \lim_{n\to\infty} \alpha_n$$

$$\leq \frac{d(x_{n+2}, x_{n+1})}{\{ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_n)\}},$$

and this is a contradiction. Hence $\lim_{n\to\infty} \alpha_n = \beta = 0$.
and if there exist $\mu \in [0, \frac{1}{2})$ such that
d(x_n, x_{n+1}) $\leq \mu$ d(x_{n-1}, x_n) $\leq \ldots \leq \mu^n$d(x_0, x_1) . . . (7)
given F_n = d(x_n, x_{n+1}) and F_{n-1} =d(x_{n-1}, x_n), and reading from (7), we have
$F_n \leq \mu F_{n-1} \leq \mu^2 F_{n-2} \leq \ldots \leq \mu^n F_0$
We now demonstrate that $\{x_n\}$ is a Cauchy sequence in X. We let

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m>n and by the inequality (iv) of Definition (2) and (1), we get
d(x_n, x_m) ≤ {d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + . . . + d(x_{n+m-1}, x_m)}
−{d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_{n+2}) + . . . + d(x_{n+m-1}, x_{n+m-1})}
≤ μ^nd(x_0, x_1) + μ^{n+1}d(x_0, x_1) + . . . + μ^{n+m-1}d(x_0, x_1)
= μ^n[d(x_0, x_1) + μd(x_0, x_1) + . . . + μ^{m-1}d(x_0, x_1)]
= $\mu^n[1 + \mu + . . . \mu^{m-1}] G_0$
Applying n, m → ∞ as d(x_n, x_m) →0, for $\mu \in [0, \frac{1}{2})$, hence $\{x_n\}$ is a Cauchy sequence in X. In addition, since (X, d) is complete,
We now by (iv) of definition 3,
d(x_n, x_m) ≤ α{ d(x_{n-1}, x_m) + d(x_m, x_n) + d(x_n, x_{n-1}) + μ d(x_{n-1}, x_n) − d(x_{n-1}, x_{n-1})
d(x_n, x_m) (1−α)≤ α{ d(x_{n-1}, x_m) + d(x_n, x_{n-1})} + μ d(x_n, x_m)
≤ $\frac{\alpha}{1-\alpha}$ { d(x_{n-1}, x_m) + d(x_n, x_{n-1})} + $\frac{\mu}{1-\alpha}$d(x_n, x_m) (8)
And as n, m → ∞, the right hand side of moves to zero.
so there exist u∈X such that x_n → u, as n→ ∞, x∈X, we have
d(Gu, u) = $\lim_{n\to\infty} d(u, x_n) = \lim_{n,m\to\infty} d(x_n, x_m) = 0$ (9)
At this point, we observe that by (9), d(Gu, u) =0, we are required to demonstrate that u is a fixed point of G. By (ii) of definition (3), we have
d(Gu, Gu)≤ d(Gu, u)
and since d(Gu, u) = 0 means d(Gu, Gu) =0 as d(u, x) =0. Thus, we have
d(Gu, Gu)= d(Gu, u) = d(u, u)
so we have Gu = u by (i) of definition 3. Hence, u is a fixed point of G.

Uniqueness: Let v be another fixed point of T with u≠v, we have
d(u, v) = d(Gu, Gv)
≤ α{d(u, v) +d(u, Gu) +(v, Gv)}
As a result
d(u, v) ≤ μd(u, v)
Implies
d(u, v) = 0
As $\mu \in [0, \frac{1}{2})$
Which implies that u=v, thus the fixed point of G, is unique.

Example 3: Given E = {3, 5, 7} and d: E×E → [0, ∞) be defined by
d(3, 3) = d(5, 5) = 0, d(7, 7) = $\frac{1}{5}$
- d(1, 3) = d(3, 1) = $\frac{1}{4}$
- d(1, 5) = d(5, 1) = 3
- d(5, 3) = d(3, 5) = 6

where we have d(u, u)<d(u, v), ∀ u, v ∈ E

1. d(1, 3) ≤ d(1, 5) + αd(5, 3) – d(5, 5), ∀ α ≥1
2. d(3, 1) ≤ d(3, 5) + αd(5, 3) – d(5, 5), ∀ α ≥1
3. d(1, 5) ≤ d(1, 3) + αd(3, 5) – d(3, 3), ∀ α ≥1
4. d(5, 1) ≤ d(5, 3) + αd(3, 1) – d(3, 3), ∀ α ≥1
5. d(5, 3) ≤ d(5, 1) + αd(1, 3) – d(1, 1), ∀ α ≥ 12
6. d(3, 5) ≤ d(3, 1) + αd(1, 5) – d(1, 1), ∀ $\alpha \geq \frac{23}{12}$

The result indicate (E, α, d) is a SPb MS, where α =12, but it is neither strong b metric nor metric space as d(5, 5) = $\frac{1}{5}$ ≠ 0.
So, the above cannot be applied to theorem (), therefore let's T : E → E be a self map defined by T1= 1, T3 =1, T5 =3
and λ ∈G defined by
μ(x) = $\frac{1}{3}\sqrt[2]{2^{-\frac{x}{10}}}$
for x>0 and $\mu(0) \in [0, \frac{1}{3})$
then
- d(T1, T3) = d(0, 0) = 0 < 0.1652 = $\frac{1}{6}\sqrt[2]{2^{-\frac{1}{40}}}$ = μ(d(1, 3)){d(1, T1) + d(3, T3)+ d(1, 3)}
- d(T1, T5) = d(1, 3) = $\frac{1}{4}$ < 2.7037 = 3$\sqrt[2]{2^{-\frac{3}{10}}}$ = μ(d(1, 5)){d(1, T1) + d(5, T5)+d(1, 5)}
- d(T3, T5) = d(1, 3) = $\frac{1}{4}$ < 3.3167 = $\frac{49}{12}$ $\sqrt[2]{2^{-\frac{6}{10}}}$ = μ(d(3, 5)){d(3, T3) + d(5, T5)+d(3, 5)}

therefore, we have S meeting all the conditions of theorem (4) and has a fixed point u =0.

Example 4. Given T : E → E and v, u ∈ [0, $\frac{1}{2}$[,we have

$$Tv = \begin{cases} \frac{v}{13} \ , if \ \ v \in [0, \ \frac{1}{2}[\\ \frac{1}{11} \ , if \ \ v = \ \frac{1}{2} \end{cases}$$

Let v, u ∈ [0, $\frac{1}{2}$[. Thus
$|Tv - Tu| = \left|\frac{v}{13} - \frac{u}{13}\right| = \frac{1}{13}|v - u|$
and
$|v - Tv| = \left|\frac{v}{13} - v\right| = \frac{12}{13}v, \quad |u - Tu| = \frac{12}{13}u$ and $|v - u| =$ v−u

which implies that
$|Tv - Tu| = \lambda\{|v - Tv| + |u - Tu| + |v - u|\}$
$= \{\frac{12}{13}v + \frac{12}{13}u$ + v−u }
$= \{ \frac{v}{13} + \frac{u}{13}\} = \frac{1}{13}$(v+u)
$|Tv - Tu| = \frac{1}{13}|v - u| \leq \frac{1}{13}$(v+u)
now, if v ∈ [0, $\frac{1}{2}$[and u = $\frac{1}{2}$, we get
$|Tv - Tu| = \left|\frac{v}{13} - \frac{1}{11}\right|$
$|v - Tv| = \frac{12}{13}v, \quad |T1 - 1| = \frac{10}{11}, |v - u| = |v - 1)$
Consequently, we have v, u ∈ [0, $\frac{1}{2}$], and thus
$|Tv - Tu| \leq \frac{v}{13} + \frac{1}{11} \leq \frac{1}{13}$ ($|v - Tv| + |u - Tu| + |v - u|$)

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