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MODIFIED LAPLACE-VARIATIONAL ITERATION METHOD FOR SOLVING LINEAR AND NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

This study presents a modified Laplace-variational iteration method (MLVIM) designed to solve linear and nonlinear Volterra integro-differential equations (VIDEs) with specified initial conditions. The MLVIM is a hybrid approach that integrates the strengths of the Laplace transform and the variation iteration method (VIM), effectively enhancing the overall solution process by improving both the efficiency and convergence rate. Specifically, the method refines the correction functional and optimizes the handling of the integral term, which directly leads to a reduction in the number of iterations needed and decreases the associated computational complexity. To demonstrate the effectiveness of MLVIM, the study applies it to two illustrative examples involving both linear and nonlinear VIDEs, with initial conditions. The results are then compared to those obtained using the Adomian decomposition method (ADM) and the fourth-order Runge-Kutta (RK4) algorithm. The findings show that MLVIM consistently exhibits a faster convergence rate and higher accuracy compared to both ADM and RK4 in all the examples presented. The MLVIM can be applied to a broad range of linear and nonlinear VIDEs. This makes it a valuable tool with potential applications in various scientific and engineering fields, where integro-differential equations frequently arise in modeling complex systems and processes.

Keywords: Volterra integro-differential equations, linear, nonlinear, Laplace transform, variational iteration method.

INTRODUCTION

Integro-differential equations is an important branch of modern Mathematics often encountered in various applied disciplines such as engineering, electrostatics, mechanics, elasticity theory, and mathematical physics (Hamoud and Ghadle (2018). These equations can be generated in two main ways. One method is by converting an initial value problem (IVP) with prescribed initial conditions, into an integro-differential form. Alternatively, they can also arise from boundary value problems (BVPs), where the equation is defined with certain boundary conditions. It is important to point out that transforming initial value problems into integral equations, or vice versa, is a well-established practice in literature. This method simplifies the process of solving these equations. However, converting boundary value problems into integral equations, or reversing that process, is less commonly explored in mathematical practice (Burton, 2005; Brunner, 2017).

Integro-differential equations arise when a differential equation involves a combination of derivatives and integrals of unknown functions. When real-life phenomena are modelled and analyzed, they usually result in functional differential equations such as partial differential equations, integral and integro-differential equations, stochastic equation etc. Integral and integro-differential equations are mathematical formulations that serve as models for a wide range of practical scenarios. In applied science and engineering, integro-differential equations are very useful. Integro-differential equations are a natural way to characterize models originating from challenges studied in fluid dynamics, ecology, biology, networking analysis, viscoelasticity, chemical kinematics, and financial matters (Cushing, 1977, Constantin, 1996; Christensen, 2003). The Italian mathematician Vito Volterra (1860–1940) created a particular kind of integro–differential equation known as Volterra integro–differential equations. In the early 1900s, Volterra investigated the effects of heredity while looking at a population growth model. The investigation produced a particular issue wherein integral and differential operators coexist in one equation. The Volterra integro-differential equations are a novel class of equations that bear his name. Regarding the theory of integral equations and their applications, Volterra made important contributions. Integral and differential equations are combined to create Volterra integro-differential equations. A general nth −order integro-differential equation is given in the form:

$$u^{(n)}(t) = f(t) + \lambda \int_0^t k(t,r)u(r)dr, \quad (1)$$

where $u^{(n)}(t) = \frac{d^n(t)}{dt^n}$, is the nth derivative with $u(t)$ being an unknown function, $k(t,r)$ is the kernel, $\mathrm{f(t)}$ is a given real-valued function, λ is a parameter and $\mathrm{u(r)}$ is the unknown function.

These kinds of problems have already been solved using a variety of analytical and numerical techniques (Wazwaz 2010; 2011). Numerous fields have benefited from the use of these equations, including population dynamics, heat transport, glass making, dynamic systems, and mathematical biology. Gribenberg *et al* (1985). The Volterra integro-differential equations, both linear and nonlinear, have garnered increasing attention in recent times. Many disciplines of nonlinear functional analysis and its applications in the science of engineering, mechanics, physics, electrostatics, biology, chemistry, and economics rely heavily on the nonlinear Volterra integro-differential equations (Abbaoui and Cherruault, 1994). Integro-differential equations have been the subject of extensive investigation by a number of researchers in recent years. It is necessary to find approximate or numerical solutions to these equations because they are typically challenging to solve and difficult to find their exact solutions. Consequently, research efforts have concentrated on creating more advanced and effective analytical and numerical techniques for resolving integro-differential equations. These techniques include the wavelet-Galerkin–method (Kumar and Mehra, 2006; Suk-in and Schulz,

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2013), Adomain decomposition method (Adomian, 1998; Wazwaz, 1999; Batiha *et al.,* 2008; Heidarzadeh *et al.,* 2012); Homotopy analysis method (He, 2004); He (1999) produced the Homotopy perturbation technique of solution, Hemeda (2012) presented a new Iterative Method of application to nth-Order integro-differential equations and lot more.

The Laplace transform is a widely used method in Mathematics and Engineering to solve differential equations. It converts a function of a real variable t (often time) to a function of a complex variable. The method transforms a time-domain function into an s-domain function, simplifying the process of solving linear differential equations (Ingham, 2013).

The variational iteration method (VIM) Proposed by Ji-Huan He is an iteration method for finding approximate solutions to various types of differential equations, including nonlinear ones. The method uses variational theory to construct correction functional, iteratively refining the solution (He, 1999).

In recent years, combining these methods in solving integro-differential equations has been an interesting study. Njoseh (2016) presented a numerical method called the Variation iteration Adomian decomposition method (VIADM) for solving nonlinear partial differential equations (PDEs) and opined that the method modifies the traditional formulation of the variation iteration decomposition method (VIDM) such that it converges more rapidly to the analytic solution. In 2019, Hamound et-al applied the modified Laplace Adomian decomposition method to find the approximate solutions of Caputo fractional Volterra-Fredholm integro-differential equation and asserted that the reliability of the method and reduction in the size of the computational work gave the method a wider applicability.

The Laplace variational iteration method (LVIM) combines the concepts and methodologies of two significant Mathematical techniques: Laplace transforms and variational iteration method to solve VIDES. The process involves applying the Laplace transform to the integro-differential equation, solving the resulting algebraic equation using VIM, and then applying the inverse Laplace transform to obtain the solution in the time domain. The initial approximation is defined and solving iteratively, the approximate solution is obtained. The combination of these approaches provides a robust framework that enables the efficient solution of complex Volterra integro-differential equations.

In mathematics, Laplace transform based numerical methods play a major role in the field of computational and applied mathematics. To make it iterative, variational methods have attracted the attention of various researchers and scientists. Combination of these two methods gives numerical results with significant convergence. Gurpreet and Inderdeep (2020) presented a semi analytical method for solving three- dimensional diffusion and wave equations arising in several applications of engineering. The proposed technique was based on the combination of Laplace transform and modified variational iterative method. The results obtained shows that the numerical technique based on the combination of two well-known numerical methods such as Laplace transform method and variational iteration method, is a powerful semi analytical technique for solving three- dimensional heat and wave equations. Laplace variational iteration strategy has been adopted to find the solution of differential equations, time fractional diffusion equations arising in porous medium, nonlinear partial differential equations, homogeneous Smoluchowski coagulation equation arising in engineering, modified fractional derivatives with non-singular kernel and numerical solutions of a family of Kuramoto-Sivashinsky equations.

Ibrahim *et al* (2021) derived and applied a new approach by modifying Laplace transform variational iteration method to solve fourth-order fractional integro-differential equations. Nonlinear boundary value problems for fourth-order fractional integro-differential equations were solved by modified Laplace transform variational iteration method. This modification is designed to improve the accuracy of the approximate solutions obtained by LVIM. In this research work, we combined the Laplace transform and the variation method (VIM) with some modifications to solve linear and nonlinear volterra integro-differential equations.

The study of Volterra integro-differential equations (VIDEs) has garnered significant attention due to their widespread application in various scientific and engineering fields, such as biological systems, population dynamics, and fluid mechanics. These equations, which combine integral and differential terms, are often complex and challenging to solve, especially in the nonlinear case. Over time, several analytical and numerical methods have been proposed to address this challenge.

However, significant gaps remain, particularly in addressing nonlinear VIDEs, improving convergence rates, handling complex boundary conditions, and developing more computationally efficient methods. The Modified Laplace-Variational Iteration Method (MLVIM) seeks to address these gaps by combining the strengths of the Laplace transform and VIM, offering a more robust and efficient approach to solving both linear and nonlinear VIDEs.

MATERIALS AND METHODS

Laplace Transform and Its Properties

Laplace Transform: Let $g(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $g(t)$, represented as $L\{g(t)\}$ is defined as:

$$L\{g(t)\} = \int_0^\infty e^{-sx} g(t)dt = \bar{g}(s) \tag{2}$$

provided the integral exists and $'s'$ is a parameter which may be a real or complex number.

Therefore

$$L\{g(t)\} = \bar{g}(s) \tag{3}$$

that is

$$g(t) = L^{-1}\{\bar{g}(s)\} \tag{4}$$

The term $L^{-1}\{\bar{g}(s)\}$, is called the inverse Laplace transform of $\bar{g}(s)$

Linearity: Let $f(t), g(t)$ be two functions of t defined for all positive values of t.Then

$$L\{a.f(t) + b.g(t)\} = a.L\{f(t)\} + b.L\{g(t)\} \tag{5}$$

where a and b are arbitrary constants.

Differentiation: Let $f(t)$ be a function of t defined for all positive values of t. Then, the Laplace transform of n^{th} derivative of function $f(t)$ is

$$\mathcal{L}\left[\frac{d^n(f(t))}{dt^n}\right] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{n-1}(0) \tag{6}$$

Inverse Laplace transforms: Let $f(t), g(t)$ be two functions of t defined for all positive values of t. Let $\bar{f}(s)$ and $\bar{g}(s)$ be the functions of s such that $\bar{f}(s) = L\{f(t)\}$ and $\bar{g}(s) = L\{g(t)\}$.

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Then

$$L^{-1}\{c.\bar{f}(s) + d.\bar{g}(s)\} = c.L^{-1}\{\bar{f}(s)\} + d.L^{-1}\{\bar{g}(s)\} = c.f(t) + d.g(t) \quad (7)$$

where c and d are arbitrary constants.

Variational Iteration Method

Variational iteration method is a most powerful technique for solving linear and nonlinear differential equations. This technique is widely used to evaluate the exact or approximate solutions of linear or nonlinear problems. The variational iteration method gives the solution in a rapidly convergent infinite series.

Consider the following general nonlinear problem:

$$Lu(t) + Nu(t) = g(t) \quad (8)$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytical function. The variational iteration method (He, 1999) constructs an iterative sequence called the correction functional as:

$$u_{n+1}(t) = u_n(t) + \int_0^x \lambda(\tau)\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \quad (9)$$

where $u_0(t)$ initial approximation with possible unknowns, λ is the general Lagrange multiplier which can be identified optimally via the variational theory, $\tilde{u}_n(\tau)$ is considered as the restricted variation, i.e. $\delta\tilde{u}_n(\tau) = 0$, and the index n denotes the n^{th} iteration. The successive approximation can be established by determining a general Lagrange multiplier λ given as

$$\lambda_n(s) = \frac{(-1)^n(s-t)^{n-1}}{(n-1)!} \quad (10)$$

The successive approximations $u_k(t,r), n \geq 0$ of the solution $u(t,r)$ are obtained on using the computed Lagrange multiplier and any selective function u_0. The exact solution may be obtained by

$$u(t,r) = \lim_{n\to\infty} u(t,r) \quad (11)$$

And the error formulation for the problem is defined as

$$\text{Absolute error} = |u(t) - u_n(t)| \quad (12)$$

The Modified Laplace-Variational Iteration Method

Consider the Volterra integro-differential equation of the form

$$u^{(n)}(t) = f(t) + \int_0^t k(t,r)u(r)dr \quad (13)$$

where $u^{(n)}(t) = \frac{d^n t}{dt^n}$, $k(t,r)$ is the kernel, $f(t)$ is a given real-valued function and $u(0), u'(0), u''(0), \dots, u^{i-1}(0)$ are the initial conditions.

Take Laplace transform of both sides of (13) to obtain

$$\mathcal{L}\{u^{(n)}(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\} \quad (14)$$

$$s^n\mathcal{L}[u(t)] - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - su^{(n-2)}(0) - u^{n-1}(0) = \mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\} \quad (15)$$

$$s^n u(s) - \sum_{i=0}^{\infty} s^{n-k-1}u^k(0) = \mathcal{L}\{f(t)\} + \mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\} \quad (16)$$

$$u(s) = \frac{1}{s^n}\sum_{i=0}^{\infty} s^{n-k-1}u^k(0) + \frac{1}{s^n}\mathcal{L}\{f(t)\} + \frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\} \quad (17)$$

By taking the inverse Laplace transform of both sides, we have

$$u(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\}\right\} \quad (18)$$

where

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\sum_{i=0}^{\infty} s^{n-k-1}u^k(0) + \frac{1}{s^n}\mathcal{L}\{f(t)\}\right\},$$

represents the terms arising from the source term and the prescribed initial conditions. Using the approach of Tarig *et al* (2013); we have that the first derivative of equation (18) is given by

$$\frac{du(t)}{dt} = \frac{dg(t)}{dt} + \frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u(r)dr\right\}\right\} \quad (19)$$

By taking the correction functional of (19), we have

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{\left(\frac{d}{d\rho}u_n(\rho) - \frac{d}{d\rho}g(\rho) - \frac{d}{d\rho}\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^\rho k(\rho,r)u_n(r)dt\right\}dr\right\}\right\} d\rho \quad (20)$$

From (20), we determine the Lagrange multiplier as

$$\lambda(s) = -1$$

Hence we have

$$u_{n+1}(t) = u_n(t) - \int_0^t \left\{\left(\frac{d}{d\rho}u_n(\rho) - \frac{d}{d\rho}g(\rho)\right. - \frac{d}{d\rho}\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^\rho k(\rho,r)u_n(r)dt\right\}\right\}dr\right\} d\rho \quad (21)$$

From (21), we have

$$u_{n+1}(t) = u_n(t) - \left\{u_n(t) - g(t) - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u_n(r)dr\right\}\right\}\right\} \quad (22)$$

Then the correction functional for the modified VIM is given by

$$u_{n+1}(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u_n(r)dr\right\}\right\}, \quad n \geq 0 \quad (23)$$

Making $g(t)$ the initial guess, we have that the initial conditions $u_0 = g(t)$

The recurrence relation is given as

$$u_0 = g(t)$$

When $n = 0$

$$u_1(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u_0(r)dr\right\}\right\}$$

When $n = 1$

$$u_2(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u_1(r)dr\right\}\right\}$$

When $n = 2$

$$u_3(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^t k(t,r)u_2(r)dr\right\}\right\}$$

Generally,

$$u_n(t) = g(t) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\mathcal{L}\left\{\int_0^t k(t,r)u_{n-1}(r)dr\right\}\right\}$$

Solving iteratively, we obtain the solution in the form

$$u(t) = \lim_{n\to\infty} u_n(t) \quad (24)$$

RESULTS

In this section, we implement the MLVIM in solving Volterra integro-differential equations. The method is compared with Adomian decomposition method (ADM) and Runge-Kutta4 algorithm for efficiency and convergence. We consider problems that have analytic solutions in order to be able to obtain the error estimates and rates of convergence for each method.

Example 4.1: (Kekena *et al.*, 2015)

Consider the following initial value problem for the third order linear Volterra integro-differential equation

$$\begin{cases} u'''(t) = 1 + t + \frac{t^3}{6} + \int_0^t (t-r)u(r)dr \\ u(0) = u''(0) = 1, u'(0) = 2 \end{cases} \quad (25)$$

The exact solution of (25) is $u(t) = e^t - t$.

By the LVIM, we take the Laplace transform of both sides of the equation in (25) to get

$$\mathcal{L}\{u'''(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{t\} + \mathcal{L}\left\{\frac{t^3}{6}\right\} + \mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\} \quad (26)$$

This gives

$$s^3u(s) - s^2u(0) - su'(0) - u''(0) = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^4} + \mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\} \quad (27)$$

At $u(0) = u''(0) = 1$, and $u'(0) = 0$, we have

$$s^3u(s) - s^2 - 1 = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^4} + \mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\} \quad (28)$$

$$u^3(s) = 1 + s^2 + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^4} + \mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\} \quad (29)$$

$$u(s) = \frac{1}{s} + \frac{1}{s^3} + \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^7} + \frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\} \quad (30)$$

Taking the inverse Laplace we have

$$u(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\}\right\} \quad (31)$$

By differentiating equation (31) above, we have

$$\frac{du(t)}{dt} = \frac{d}{dt}(1) + \frac{d}{dt}\left(\frac{t^2}{2!}\right) + \frac{d}{dt}\left(\frac{t^3}{3!}\right) + \frac{d}{dt}\left(\frac{t^4}{4!}\right) + \frac{d}{dt}\left(\frac{t^6}{6!}\right) + \frac{d}{dt}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u(r)dr\right\}\right\}\right\} \quad (32)$$

Next, we define the correction functional of equation (32)

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{\left(\frac{d}{d\rho}u_n(\rho) - \frac{d}{dt}(1) - \frac{d}{dt}\left(\frac{\rho^2}{2!}\right) - \frac{d}{dt}\left(\frac{\rho^3}{3!}\right) - \frac{d}{dt}\left(\frac{\rho^4}{4!}\right) - \frac{d}{dt}\left(\frac{\rho^6}{6!}\right) - \frac{d}{dt}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^\rho (t-r)u_n(r)dr\right\}\right\}dr\right\} d\rho \quad (33)$$

From equation (33) we determine the Lagrange multiplier as

$$\lambda(s) = -1$$

Hence we have

$$u_{n+1}(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u_n(r)dr\right\}\right\} \quad (34)$$

where

$$u_0(t) = g(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!}$$

$$u_1(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u_0(r)dr\right\}\right\}$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)\left(1 + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^6}{6!}\right)dr\right\}\right\}$$

$$u_1(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \frac{t^{11}}{11!}$$

$$u_2(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u_1(r)dr\right\}\right\}$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)\left(1 + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \frac{r^6}{6!} + \frac{r^7}{7!} + \frac{r^8}{8!} + \frac{r^9}{9!} + \frac{r^{11}}{11!}\right)dr\right\}\right\}$$

$$u_2(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \frac{t^{10}}{10!} + \frac{t^{11}}{11!} + \frac{t^{12}}{12!} + \frac{t^{13}}{13!} + \frac{t^{14}}{14!} + \frac{t^{16}}{16!}$$

$$u_3(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u_2(r)dr\right\}\right\}$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\int_0^t (t-r)\left(1 + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \frac{r^6}{6!} + \frac{r^7}{7!} + \frac{r^8}{8!} + \frac{r^9}{9!} + \frac{r^{10}}{10!} + \frac{r^{11}}{11!} + \frac{r^{12}}{12!} + \frac{r^{13}}{13!} + \frac{r^{14}}{14!} + \frac{r^{16}}{16!}\right)dr\right\}\right\}$$

$$u_3(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \frac{t^{10}}{10!} + \frac{t^{11}}{11!} + \frac{t^{12}}{12!} + \frac{t^{13}}{13!} + \frac{t^{14}}{14!} + \frac{t^{15}}{15!} + \frac{t^{16}}{16!} + \frac{t^{17}}{17!} + \frac{t^{18}}{18!} + \frac{t^{19}}{19!} + \frac{t^{21}}{21!}$$

$$u_4(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^t (t-r)u_3(r)dr\right\}\right\}$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left\{\int_0^t (t-r)\left(1 + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \frac{r^5}{5!} + \frac{r^6}{6!} + \frac{r^7}{7!} + \frac{r^8}{8!} + \frac{r^9}{9!} + \frac{r^{10}}{10!} + \frac{r^{11}}{11!} + \frac{r^{12}}{12!} + \frac{r^{13}}{13!} + \frac{r^{14}}{14!} + \frac{r^{15}}{15!} + \frac{r^{16}}{16!} + \frac{t^{17}}{17!} + \frac{t^{18}}{18!} + \frac{t^{19}}{19!} + \frac{t^{21}}{21!}\right)dr\right\}\right\}$$

$$u_4(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \frac{t^{10}}{10!} + \frac{t^{11}}{11!} + \frac{t^{12}}{12!} + \frac{t^{13}}{13!} + \frac{t^{14}}{14!} + \frac{t^{15}}{15!} + \frac{t^{16}}{16!} + \frac{t^{17}}{17!} + \frac{t^{18}}{18!} + \frac{t^{19}}{19!} + \frac{t^{20}}{20!} + \frac{t^{21}}{21!} + \frac{t^{22}}{22!} + \frac{t^{23}}{23!} + \frac{t^{24}}{24!} + \frac{t^{26}}{26!}$$

Example 4.2: (Kekena *et al.*, 2015)
Consider the following initial value problem for the first order nonlinear Volterra integro-differential equation

$$\begin{cases} u'(t) = 1 + \int_0^t u^2(r)dr \\ u(0) = 0 \end{cases} \quad (35)$$

By the LVIM, we take the Laplace transform of both sides of the equation in (35) to get

$$\mathcal{L}\{u'(t)\} = \mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^t u^2(r)dr\right\} \quad (36)$$

This gives

$$su(s) - u(0) = \frac{1}{s} + \mathcal{L}\left\{\int_0^t u^2(r)dr\right\} \quad (37)$$

At $u(0) = 0$, we have

$$su(s) = \frac{1}{s} + \mathcal{L}\left\{\int_0^t u^2(r)dr\right\} \quad (38)$$

$$u(s) = \frac{1}{s^2} + \frac{1}{s}\mathcal{L}\left\{\int_0^t u^2(r)dr\right\} \quad (39)$$

Taking the inverse Laplace we have

$$u(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u^2(r)dr\right\}\right\} \quad (40)$$

By differentiating equation (40) above, we have

$$\frac{du(t)}{dt} = \frac{d}{dt}(t) + \frac{d}{dt}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u^2(r)dr\right\}\right\}\right\} \quad (41)$$

Next, we define the correction functional of equation (41)

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{(\frac{d}{d\rho}u_n(\rho) - \frac{d}{dt}(\rho) - \frac{d}{dt}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\{\int_0^\rho u^2(r)dr\}\right\}\right\}dr\right\} d\rho \quad (42)$$

From equation (42) we determine the Lagrange multiplier as

$$\lambda(s) = -1$$

Hence we have

$$u_{n+1}(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u_n^{\,2}(r)dr\right\}\right\} \quad (43)$$

where

$$u_0(t) = g(t) = t$$

$$u_1(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u_0^{\,2}(r)dr\right\}\right\}$$

$$= t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t r^2dr\right\}\right\}$$

$$u_1(t) = t + \frac{t^4}{12}$$

$$u_2(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u_1^{\,2}(r)dr\right\}\right\}$$

$$u_2(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t \left(r + \frac{r^4}{12}\right)^2 dr\right\}\right\}$$

$$u_2(t) = t + \frac{t^4}{12} + \frac{t^7}{252} + \frac{t^{10}}{12960}$$

$$u_3(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u_2^{\,2}(r)dr\right\}\right\}$$

$$= t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t \left(r + \frac{r^4}{12} + \frac{t^7}{252} + \frac{t^{10}}{12960}\right)^2 dr\right\}\right\}$$

$$u_3(t) = t + \frac{t^4}{12} + \frac{t^7}{252} + \frac{t^{10}}{6048} + \frac{37t^{13}}{7076160} + \frac{109t^{16}}{914457600} + \frac{29t^{19}}{197990072280} + \frac{t^{22}}{77598259200}$$

$$u_4(t) = t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t u_3^{\,2}(r)dr\right\}\right\}$$

$$= t + \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\int_0^t \left(r + \frac{r^4}{12} + \frac{t^7}{252} + \frac{t^{10}}{12960} + \frac{37t^{13}}{7076160} + \frac{109t^{16}}{914457600} + \frac{29t^{19}}{197990072280} + \frac{t^{22}}{77598259200}\right)^2 dr\right\}\right\}$$

$$u_4(t) = t + \frac{t^4}{12} + \frac{t^7}{252} + \frac{t^{10}}{6048} + \frac{t^{13}}{157248} + \frac{2663t^{16}}{11887948800} + \frac{4799t^{19}}{677613081600} + \frac{34109t^{22}}{170758496563200} + \frac{4507t^{25}}{901604861853696} + \frac{24354871t^{28}}{221524314557453107200} + \frac{1312457t^{31}}{628869391142952960000} - \frac{253524431t^{34}}{764770095179884265547200000} + \frac{1709t^{37}}{405316465646329036800} + \frac{241247t^{40}}{599430189193706078208000000} + \frac{t^{43}}{391328412985769656320000} + \frac{t^{46}}{124644839499016962048000000}$$

Table 4.1a: Comparison of Results of example 4.1

T	exact solution	MLVIM at u_2	u_{ADM}	u_{RK4}
0	1	1	1	1
0.1	1.005170918	1.005170918	1.005170918	1.005170918
0.2	1.021402758	1.021402758	1.021402758	1.021402758

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0.3	1.049858808	1.049858808	1.049858808	1.049858808
0.4	1.091824698	1.091824698	1.091824698	1.091824698
0.5	1.148721271	1.148721271	1.148721271	1.148721271
0.6	1.2221188	1.2221188	1.2221188	1.2221188
0.7	1.313752707	1.313752707	1.313752707	1.313752707
0.8	1.425540928	1.425540928	1.425540928	1.425540928
0.9	1.559603111	1.559603111	1.559603111	1.559603111
1	1.718281828	1.718281828	1.718281828	1.718281828

Table 4.1b: **Error analysis of example 4.1**

MLVIM Error	u_{ADM} Error	u_{RK4} Error
0.00E+00	0.00E+00	0.00E+00
0.00E+00	8.00E-11	8.00E-11
0.00E+00	1.60E-10	1.60E-10
0.00E+00	-4.20E-10	-4.20E-10
0.00E+00	-3.60E-10	-3.60E-10
0.00E+00	-3.00E-10	-3.00E-10
0.00E+00	3.90E-10	3.90E-10
0.00E+00	4.70E-10	4.70E-10
0.00E+00	4.90E-10	4.90E-10
0.00E+00	1.60E-10	1.60E-10
0.00E+00	4.60E-10	4.60E-10

Table 4.2a: Comparison of Results of example 4.2

t	exact solution	MLVIM at u_3	u_{ADM}	u_{ADM}
0.1	0.1000083	0.10000834	0.1000083	0.1000083
0.2	0.2001334	0.20013338	0.2001334	0.2001334
0.3	0.3006759	0.30067587	0.3006759	0.3006759
0.4	0.4021399	0.40213985	0.4021399	0.4021399
0.5	0.5052395	0.5052395	0.5052395	0.5052395
0.6	0.6109121	0.61091209	0.6109121	0.6109121
0.7	0.7203399	0.72033987	0.7203399	0.7203399
0.8	0.8349836	0.83498365	0.8349837	0.8349837
0.9	0.9566323	0.95663232	0.9566323	0.9566323
1	1.0874733	1.08747353	1.0874735	1.0874735

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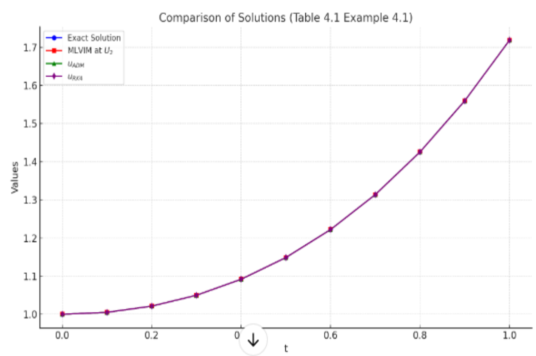
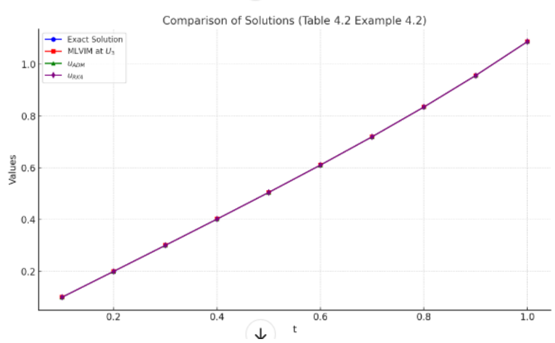
Table 4.3b: Error analysis of example 4.3

MLVIM Error	u_{ADM} Error	u_{RK4} Error
-3.730E-08	-3.000E-08	-3.000E-08
1.586E-08	2.000E-08	2.000E-08
3.117E-08	3.000E-08	3.000E-08
4.770E-08	5.000E-08	5.000E-08
2.430E-09	0.000E+00	0.000E+00
6.140E-09	1.000E-08	1.000E-08
3.093E-08	3.000E-08	3.000E-08
-4.648E-08	-5.000E-08	-5.000E-08
-1.628E-08	-2.000E-08	-2.000E-08
-2.323E-07	-2.000E-07	-2.000E-07





DISCUSSION OF RESULTS

The resulting numerical evidences from Examples 1 and 2 show that the method is reliable and accurate with excellent convergence rate as illustrated in the **graphs** and **tables**. The results obtained were compared with those available in literature and has proven to be very accurate and efficient from its application in the examples. In Example 4.1, third order linear VIDE was solved using the MLVIM and the result obtained was compared with the exact solution, ADM and RK4. The ADM and the RK4 produced some error while the MLVIM converges absolutely to the exact solution. In Example 4.2, the MLVIM was applied to a first order nonlinear VIDE with initial condition. The results obtained from the MLVIM when compared with the exact solution, ADM and the RK4 converges to the exact solution and minimal error was obtained. The MLVIM has shown to be accurate, efficient and reliable in all the examples considered as it reduces the number of iterations and converge faster to the exact solution when compared with results in literature. The method can be used to solve any linear or nonlinear Volterra integro differential equations.

Conclusion

This research has provided a comprehensive overview of the Modified Laplace variational iteration method and its application in solving Volterra integro-differential equations. Some examples were considered and presented in tabular and graphical formats in order to compare the MLVIM with exact solution and other existing methods in literature and the method has shown to be very accurate as it reduces the number of iterations and converges rapidly to the exact solution. All computational frameworks are implemented by MATLAB software.

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