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Full Length Research Article

MATHEMATICAL ANALYSIS OF MPOX MODEL IN THE PRESENCE OF EARLY SCREENING WITH THERAPY AND ISOLATION WITH TREATMENT AS CONTROL STRATEGIES

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ABSTRACT
This paper developed and analyzed a mathematical model of Mpox infection, incorporating early screening with therapy of the exposed and isolation with treatment of the infected individuals. The existence and uniqueness of the model solution, positivity, and the feasible region of the model solution were shown. Two equilibrium states of the model, namely disease-free equilibrium (DFE) state and endemic equilibrium (EE), were determined, and the basic reproduction number R_0, was calculated using the next generation matrix method. It was shown that the DFE state is locally and globally asymptotically stable when $R_0 = \max(R_{0h}, R_{0n}) < 1$. Similarly, the EE state was shown to be locally and globally asymptotically stable when $R_0 = \max(R_{0h}, R_{0n}) > 1$, where $R_{0h}\ and\ R_{0n}$ is the basic reproduction number for human and animal populations, respectively. Furthermore, the results of the sensitivity analysis of the basic reproduction number with respect to the model parameters show that screening with therapy of exposed, progression from exposed class to infected class and isolation with treatment of infected individuals, are the most sensitive parameters. These analytical results would be validated using numerical simulation with real data.

Keywords: Mpox, early screening with exposure therapy, isolation, existence and uniqueness, basic reproduction number.

INTRODUCTION

In recent times, more than seventy (70) countries have been affected by the outbreak of Monkeypox(Mpox). This has led to the suggestion that the Monkeypox Virus (MPXV) should be declared a public health emergency in endemic regions (WHO, 2022).
Mpox, is a family of Orthopoxvirus, which is a Zoonotic disease that is caused by Monkeypox Virus (MPXV) (Center for Disease Control (CDC), 2003; Heskin, et al., 2022; Rahman, et al., 2020). Mpox is a viral infectious disease, which is commonly found in remote settlements, mostly in Central and West Africa region, where human come in contact with an infected non-human which is a primary channel of transmission (Jezek *et al.*, 1988; Rizk, et al., 2022; Railian, et al., 2023; Valavan & Meryer, 2022).
MPXV is transmitted through Animal to Human (A2H) transmission, Human to Human (H2H) transmission and Human to Animal (H2A) via direct or indirect contact with an infected animal/person, environment, surface, respiratory droplet, body fluid (Alkunle *et al.,* 2020; CNBC,2023; WOAH, 2023; CDC, 2022; Murphy & Ly, 2022; Titanji, et al., 2022; Beeson, et al.,2023). The latent period of MPXV is usually 4-21 days, after which the signs and symptoms of MPXV begin to manifest in the victim's body, such as lymph node enlargement, fever, myalgia, back pain, severe headache, sweating, etc. (Esshauer *et al.,* 2010; CDC-Africa, 2022; Hraib, et al.,2022; Luo & Han, 2022; Shaheen, et al.,2022).
The prevention and management of MPXV, to reduce or mitigate the spread of the outbreak is a global concern, particularly in Africa, with Sub-Saharan Africa being the most endemic region. Currently there are several efforts that are ongoing, both locally and globally to reduce or eradicate MPXV, from causing further damage to lives and livelihood. Government agencies, individuals and researchers have committed funds and resources, in creating awareness, on the existence and potential of the disease spread. Several scholars have employed different approaches in understanding the dynamics of the disease spread.
For instance, mathematical model approaches have been utilized by different scholars, to study the dynamics of MPXV with some control measures aimed at lowering the disease progression.
A base for mathematical model for transmission dynamics of Pox-Like disease was presented. In the work all baseline parameters where assume from other pox-like virus. Results reveals malnutrition of individuals, unhealthy living of individuals and immune of individuals varies in recovery rate from Mpox due to warning immune (Bhunu & Mushayabasa, 2011).
Mathematical model of Mpox infection was formulated, incorporating the exposed, vaccination susceptible individuals, unvaccinated individuals and treatment of infected individuals, to study the dynamics of the spread of Mpox with control interventions. Numerical simulations were directed towards assessing the combine impacts of vaccination for prevention of susceptible individuals and treatment of symptomatic individuals (Usman & Adamu, 2017; Emeka *et al*., 2018; Bolaji, et al., 2024).
Mathematical model of Mpox transmission dynamics was constructed, incorporating education awareness of the susceptible, public enlightenment campaign, quarantine, detection, and undetected infected individuals compartment. Simulation results were directed towards the impacts of prevention and enlightenments of susceptible and effectiveness of getting the infected individuals quarantined (Somma et al., (2019; Peter, *et al.,*2021; Idisi, et al 2023; Soni & Sinah, 2024;Olapade et al., 2024; Sefiu, et al., 2024)
A deterministic model on the spread of Mpox was formulated, incorporating prodromal and differential in stages of infected individuals and hospitalization to assess the impact of control interventions with real data. The work focused on cumulative reported cases of infected and death from September 2017 when the virus re-emerged to January 2023 in Nigeria and the Democratic Republic of Congo (DRC) (Al-Shomrani, et al., 2023; Peter, et al., 2024).
A mathematical model for the transmission of Mpox, was constructed with a view to analyzed and evaluate the effectiveness

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of optimal control strategies. The optimal control functions were focused towards control intervention to reduce the rate infection on high and low risk infected individuals, deceased individuals and vaccinated individuals (susceptible) (Singo, et al., 2024; Ikhsani, et al., 2025).

We observed that none of the research has considered incorporating early screening with therapy and Isolation with treatment.

The present study extends the work of (Bhunu & Mushayabasa, 2011;Usman & Adamu 2017; Peter, et al.,2021), to incorporate early screening with therapy and isolation with treatment, with an aim of performing analysis on the model to assess the impact of combing both control measures in the fight to mitigate or eradicate MPXV.

MATERIALS AND METHODS

Construction of Mpox Model

We formulate a deterministic model of Mpox, by dividing the population into two namely: Human and Animal (Rodents, rats etc.). The human population is divided into five sub-compartments by adopting the general SEIHR-type (S-Susceptible, E-Exposed, I-Infectious H-Isolation and R-Recover) model and the SEI (S-Susceptible, E-Exposed, I-Infectious) for Animal population.

$$N_h = S_h + E_h + I_h + H + R_h \quad (1)$$

$$N_n = S_n + E_n + I_n \quad (2)$$

The description of the model state variables and parameters are presented in Table 1.

Table 1: Description of Variable and Parameter

Variable/Parameter	Description
$N_h(t)$	The total Human Population at time t
$N_n(t)$	The total Animal Population at time t
$S_h(t)$	The total number of Susceptible humans at time t
$E_h(t)$	The total number of Exposed humans at time t
$I_h(t)$	The total number of infected humans at time t
$H_h(t)$	The total number of isolated humans at time t
$R_h(t)$	The total number of Recovered humans at time t
$S_n(t)$	The total number of Susceptible Animals at time t
$E_n(t)$	The total number of Exposed Animals at time t
$I_n(t)$	The total number of Infected Animals at time t
π_h/π_n	Recruitment rate for Humans/Animals
ρ	Proportion of exposed class Screened with therapy
$(1-\rho)$	Proportion of exposed progressing to infected class on expiration of window period
α	Progression rate
θ_1	Isolation with treatment rate of Infected Humans
θ_2	Progression rate from isolation class to recovery class
γ	Progression rate of Exposed Animal to infected Compartment
δ_{h0}/δ_{h1}	Natural death/Death rate due to Mpox of Human
δ_{n0}	Natural Death rate in Animal

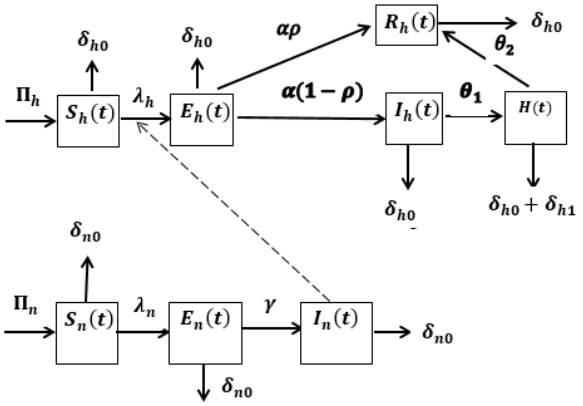

Figure 1: Schematic Description of Mpox Disease Model

Note that the force of infection due to interaction between both populations is given as:-

$$\lambda_h = \frac{\beta_h I_h}{N_h} + \frac{\beta_{nh} I_n}{N_n} \quad (3)$$

$$\lambda_n = \frac{\beta_n I_n}{N_n}$$

where the parameter(s) β_h, β_n, and β_{nh} is the transmission rate for human and Animal population.

The susceptible Human population is increased through constant recruitment rate by birth or immigration rate at Π_h and reduced by natural death rate at δ_{h0}. Hence the susceptible human population at given time t, is given as

$$\frac{dS_h(t)}{dt} = \Pi_h - (\lambda_h + \delta_{h0})S_h(t)$$

The population of the exposed human (i.e. $E_h(t)$ is achieved through interaction with susceptible human population and is reduced by natural death rate at δ_{h0} and a screened with therapy proportion rate at $\alpha\rho$ progressing to recovery class and at the expiration of latency period, the infected is generated at $\alpha(1-\rho)$ undetected during screening. Hence the exposed population at a given time$t$ is given as

$$\frac{dE_h(t)}{dt} = \lambda_h S_h(t) - (\alpha\rho + \alpha(1-\rho) + \delta_{h0})E_h(t)$$

An Infected human population is yielded as a result of interaction with exposed individuals who at the expiration of latency period progress at a rate $\alpha(1-\rho)$ and reduced by isolation with treatment rate θ_1, a natural death rate at δ_{h0} and death rate due to MPXV at rate δ_{h1}.Hence infected population at a given time t is given as

$$\frac{dI_h(t)}{dt} = \alpha(1-\rho)E_h(t) - (\theta_1 + \delta_{h0} + \delta_{h1})I_h(t)$$

The Isolated class is yielded as a result of isolation with treatment rate $\theta_1$of an infected individual and reduced by progression rate θ_2, a natural death rate at δ_{h0} and death rate due to MPXV δ_{h1}. Hence the Isolated class at given time t is given as

$$\frac{dH_h(t)}{dt} = \theta_1 I_h(t) - (\theta_2 + \delta_{h0} + \delta_h)H(t)$$

The recovered individual is yielded as a result of progression of screen with treated rate $\alpha\rho$ of exposed individuals and a progression from Isolated class at a rate θ_2 and reduced by a natural death rate at δ_{h0}. Hence the recovered human population

at a given time t is given as

$$\frac{dR_h(t)}{dt} = \alpha\rho E_h(t) + \theta_2 H_h(t) - \delta_{h0} R_h(t)$$

The susceptible animal population is increased through constant recruitment rate by birth or immigration rate at Π_n and reduced by natural death rate at δ_{n0}. Hence the susceptible animal population at given time t, is given as

$$\frac{dS_n(t)}{dt} = \Pi_n - (\lambda_n + \delta_{n0}) S_n(t)$$

The population of the exposed animal is achieved through interaction with susceptible non-human population and is reduced by natural death rate at δ_{n0} and the progression rateγ. Hence the exposed animal population at a given time$t$ is given as

$$\frac{dE_n(t)}{dt} = \lambda_h S_n(t) - (\gamma + \delta_{n0}) E_n(t)$$

The infected animal population, is yielded as a result of interaction with susceptible animal population and reduced by death rate at δ_{n0} . Hence the infected animal population at given time$t$, is given as

$$\frac{dI_n(t)}{dt} = \gamma E_n(t) - \delta_{n0} I_n(t)$$

We make the following basic assumptions for the formulation of our model as -

i. The model assumes homogeneous mixing of population, i.e. individuals have an equal chance of contacting each other.
ii. The transmission of the virus occurs through direct or indirect contact with infectious individual or Animals or contaminated surface.
iii. The rate of transmission is proportional to contact rate between individuals and animals.

With the description, flow diagram and assumptions above, the model is presented as

$$\frac{dS_h(t)}{dt} = \Pi_h - (\lambda_h + \delta_{h0}) S_h(t) \quad (4)$$

$$\frac{dE_h(t)}{dt} = \lambda_h S_h(t) - (\alpha + \delta_{h0}) E_h(t) \quad (5)$$

$$\frac{dI_h(t)}{dt} = \alpha(1-\rho) E_h(t) - (\theta_1 + \delta_{h0} + \delta_{h1}) I_h(t) \quad (6)$$

$$\frac{dH}{dt} = \theta_1 I_h(t) - (\theta_2 + \delta_{h0} + \delta_h) H(t) \quad (7)$$

$$\frac{dR_h(t)}{dt} = \alpha\rho E_h(t) + \theta_2 H_h(t) - \delta_{h0} R_h(t) \quad (8)$$

$$\frac{dS_n(t)}{dt} = \Pi_n - (\lambda_n + \delta_{n0}) S_n(t) \quad (9)$$

$$\frac{dE_n(t)}{dt} = \lambda_h S_n(t) - (\gamma + \delta_{n0}) E_n(t) \quad (10)$$

$$\frac{dI_n(t)}{dt} = \gamma E_n(t) - \delta_{n0} I_n(t) \quad (11)$$

With model initial condition as

$$S_h(0) \geq 0, E_h(0) \geq 0, I_h(0) \geq 0, H_h(0) \geq 0,\ R_h(0) \geq 0, S_n(0) \geq 0, E_h(0) \geq 0, I_n(0) \geq 0, \quad (12)$$

RESULTS AND DISCUSSION

Existence and Uniqueness of Solution for the Model

Consider the initial value problem (IVP)

$$x' = f(t,x), y(t_0) = x_0 \quad (13)$$

Whose solution exists and is unique. So that our model in (4) to (11) can be written as

$$f_1(t,x) = \Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h \quad (14)$$

$$f_2(t,x) = \beta_h \frac{I_h}{N_h} S_h + \beta_{nh} \frac{I_n}{N_n} S_h - (\alpha + \delta_{h0}) E_h(t) \quad (15)$$

$$f_3(t,x) = \alpha(1-\rho) E_h - (\theta_1 + \delta_{h0} + \delta_{h1}) I_h \quad (16)$$

$$f_4(t,x) = \theta_1 I_h - (\theta_2 + \delta_{h0} + \delta_{h1}) H \quad (17)$$

$$f_5(t,x) = \alpha\rho E_h + \theta_2 H_h - \delta_{h0} R_h \quad (18)$$

$$f_6(t,x) = \Pi_n - \beta_n \frac{I_n}{N_n} S_n - \delta_{n0} S_n \quad (19)$$

$$f_7(t,x) = \beta_n \frac{I_n}{N_n} S_n - (\gamma + \delta_{n0}) E_n \quad (20)$$

$$f_8(t,x) = \gamma E_n - \delta_{n0} I_n(t) \quad (21)$$

Theorem 1: (Lipschitz condition**),** (Boyce & Diprima,2001; Khalil, 2002)

Consider the initial value problem (IVP)

$$x' = f(t, x_1, x_2, x_3, \dots, x_n), x_1(t_0) = x_1, x_2(t_0) = x_2, x_3(t_0) = x_3, \dots, x_n(t_0) = x_n \quad (22)$$

Defined on a region $\Re$ as

$$|t - t_0| \leq a, ||x - x_0|| \leq b, (x = x_1, x_2, x_3, \dots, x_n), (x_0 = x_{10}, x_{20}, x_{30}, \dots, x_{n0}) \quad (23)$$

And suppose $f(t,x)$ satisfies the Lipschitz condition

$$||f(t,x_n) - f(t,x_{n-1})|| \leq k||x_n - x_{n-1}|| \quad (24)$$

Whenever the pair (t,x_n) and (t,x_{n-1}) are in $\Re$ and k is a positive number, then $f(t,x)$ has one solution (Uniqueness). Note that if $f(t,x)$has partial derivative $\frac{\partial f_i}{\partial x_i}$ such that $\left|\frac{\partial f_i}{\partial x_i}\right| < \infty, \forall i = 1,2,3,\dots,n$ satisfying equation (23), then the solution of $f(t,x)$ exist and bounded in region $\Re$.

From equation (14), we have

$$\frac{\partial f_1}{\partial S_h} = \frac{\partial}{\partial S_h}\left(\Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h\right) = -\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0},$$

$$\left|\frac{\partial f_1}{\partial S_h}\right| = \left|-\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0}\right| < \infty \quad , \quad \frac{\partial f_1}{\partial E_h} = \frac{\partial}{\partial E_h}\left(\Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h\right)$$ implying

$$\left|\frac{\partial f_1}{\partial E_h}\right| = 0 < \infty, \quad \frac{\partial f_1}{\partial I_h} = \frac{\partial}{\partial I_h}\left(\Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h\right),$$ then $\left|\frac{\partial f_1}{\partial I_h}\right| = \left|-\beta_h \frac{S_h}{N_h}\right| < \infty,$

$$\frac{\partial f_1}{\partial H} = \frac{\partial}{\partial H}\left(\Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h\right),$$

implying $\left|\frac{\partial f_1}{\partial H}\right| = 0 < \infty$, and $\left|\frac{\partial f_1}{\partial R_h}\right| = 0 < \infty$.

$f_1(t, S_h)$ is continuous and bounded in bounded, in the interval $0 < \Re < 1$, satisfying Lipschitz condition in equation (24). Clearly, $f_i's$ and their partial derivative, with respect to each of the state variables, follow from above. Hence there exist unique solutions of equation (4) to (11) in the region $\Re$.

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Positivity of Solution

Theorem 2: Given the initial condition$\{S_h(0), E_h(0), I_h(0), H(0), R_h(0) \geq 0\} \in \phi_h$ and $\{S_n(0), E_n(0), I_n(0) \geq 0\} \in \phi_n$ in equation (12), then the solution $\{S_h(t), E_h(t), I_h(t),, H(t), R_h(t)\}$ and $\{S_n(t), E_n(t), I_n(t)\}$ of equation (4)-(11) is nonnegative for all $t > 0$

Proof

Here, we prove that for all $t > 0$, the solution of equation (4)-(11) are always positive.

From equation (4)

$\frac{dS_h(t)}{dt} = \Pi_h - \beta_h \frac{I_h}{N_h} S_h - \beta_{nh} \frac{I_n}{N_n} S_h - \delta_{h0} S_h \geq -\left(\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0}\right) S_h$ (25)

Integrating equation (25) by separation of variables

$$\int \frac{dS_h}{S_h} \geq \int -\left(\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0}\right) dt =$$

$$\ln S_h \geq -\left(\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0}\right) t + c_1$$

Using initial condition at $t = 0$, gives

$$\Rightarrow S_h(t) \geq S_h(0) e^{-\left(\beta_h \frac{I_h}{N_h} - \beta_{nh} \frac{I_n}{N_n} - \delta_{h0}\right) t} > 0 \quad (26)$$

From equation of (5)

$\frac{dE_h(t)}{dt} = \lambda_h S_h - (\alpha + \delta_{h0}) E_h \geq -(\alpha + \delta_{h0}) E_h$ (27)

Integrating equation (27), by separation of variables, i.e. gives

$\int \frac{dE_h}{E_h} \geq \int -(\alpha + \delta_{h0}) dt = \ln E_h = -(\alpha + \delta_{h0}) t + c_2$

Using$E_h(0) = 0$ at $t = 0$, we obtain,

$E_h(t) \geq E_h(0) e^{-(\alpha + \delta_{h0}) t} > 0$ (28)

Similarly, equation (6) to (11) follows from the method used in equation (26) and equation (28), we can show the solutions for other equations are nonnegative for $t \geq 0$.

The Feasible Region

Theorem 3: Let $\phi_h = \left\{(S_h, E_h, I_h, H, R_h) \in \mathbb{R}_+^5 : N_h(t) \leq \frac{\pi_h}{\delta_{h0}}\right\}$ and

$\phi_n = \left\{(S_n, E_n, I_n) \in \mathbb{R}_+^3 : N_n(t) \leq \frac{\pi_n}{\delta_{n0}}\right\}$, so that$\phi = \left\{\phi_h \times \phi_n \middle| N_h(t) \leq \frac{\pi_h}{\delta_{h0}}; N_n(t) \leq \frac{\pi_n}{\delta_{n0}}\right\}$ then the region ϕ is positively invariant with respect to equation (4) -(11).

Proof

Adding equation (4) to (8), we have

$\frac{dN_h(t)}{dt} = \frac{dS_h}{dt} + \frac{dE_h}{dt} + \frac{dI_h}{dt} + \frac{dH}{dt} + \frac{dR_h}{dt} \Rightarrow \frac{dN_h(t)}{dt} \leq \pi_h - \delta_{h0} N_h$ (29)

We use integrating factor (IF) method as $IF = e^{\int \delta_{h0} dt} = e^{(\delta_{h0}) t}$ and Multiply both sides of equation (29), by $e^{(\delta_{h0}) t}$, and simplify we have

$$\frac{d}{dt}\left(N_h(t) e^{(\delta_{h0}) t}\right) = \pi_h e^{(\delta_{h0}) t}$$

Integrating both sides

$\int \frac{d}{dt}\left(N_h(t) e^{(\delta_{h0}) t}\right) = \int \pi_h e^{(\delta_{h0}) t} dt, \Rightarrow N_h(t) e^{(\delta_{h0}) t} = \frac{\pi_h}{\delta_{h0}} e^{(\delta_{h0}) t} + c$

Using, $N_h(0) = 0$ at $t = 0$ we have

$N_h(t) \leq N_h(0) e^{-(\delta_{h0}) t} + \frac{\pi_h}{\delta_{h0}}\left(1 - e^{-(\delta_{h0}) t}\right)$ (30)

In equation (9) to (11), we follow same method in equation (30), we obtain

$N_n(t) \leq N_n(0) e^{-(\delta_{n0}) t} + \frac{\pi_n}{\delta_{n0}}\left(1 - e^{-(\delta_{n0}) t}\right)$ (31)

From equation (30) and equation (31), in particular, if $N_h(0) \leq \frac{\pi_h}{\delta_{h0}}$ then

$N_h(t) \leq \frac{\pi_h}{\delta_{h0}}$ and if $N_n(0) \leq \frac{\pi_n}{\delta_{n0}}$ then $N_n(t) \leq \frac{\pi_n}{\delta_{n0}}$.

Hence, $= \left\{\phi_h \times \phi_n \middle| N_h(t) \leq \frac{\pi_h}{\delta_{h0}}; N_n(t) \leq \frac{\pi_n}{\delta_{n0}}\right\}$, is positively invariant,so that the solution pathway remains in the region ϕ.

Equilibrium States

We set the right-side of the model in equation (4) – (11) to zero i.e.

$$\Pi_h - \beta_h S_h \frac{I_h}{N_h} - \beta_{nh} S_h \frac{I_n}{N_n} - \delta_{h0} S_h(t) = 0 \quad (32)$$

$$\beta_h S_h \frac{I_h}{N_h} + \beta_{nh} S_h \frac{I_n}{N_n} - M_1 E_h(t) = 0 \quad (33)$$

$$M_2 E_h(t) - M_3 I_h(t) = 0 \quad (34)$$

$$\theta_1 I_h(t) - M_4 H(t) = 0 \quad (35)$$

$$\alpha\rho E_h(t) + \theta_2 H_h(t) - \delta_{h0} R_h(t) = 0 \quad (36)$$

$$\Pi_n - \beta_n S_n \frac{I_n}{N_n} - \delta_{n0} S_n(t) = 0 \quad (37)$$

$$\beta_n S_n \frac{I_n}{N_n} - (\gamma + \delta_{n0}) E_n(t) = 0 \quad (38)$$

$$\gamma E_n(t) - \delta_{n0} I_n(t) = 0 \quad (39)$$

Where $M_1 = (\alpha + \delta_{h0}), M_2 = \alpha(1 - \rho), M_3 = (\theta_1 + \delta_{h0} + \delta_{h1}), M_4 = (\theta_2 + \delta_{h0} + \delta_{h1})$ (40)

Disease-Free Equilibrium (DFE) State

At the disease free-equilibrium (DFE) state(i.e. absence of an infected individual from the community), in particular $E_h(t) = I_h(t) = I_n(t) = 0$, in equation(32)-(39), i.e.

Since $I_h = I_n = 0$ then $E_h = H = R_h = E_n = 0$, So that the DFE state of equation (9) -(11) can be written as

$$E_0 = (S_h^*, E_h^*, I_h^*, H^*, R_h^*, S_n^*, E_n^*, I_n^*) = \left(\frac{\Pi_h}{\delta_{h0}}, 0, 0, 0, 0, \frac{\Pi_n}{\delta_{n0}}, 0, 0\right) \quad (41)$$

Endemic Equilibrium (EE) State

For endemic equilibrium state i.e. $I_h \neq 0$ and $I_n \neq 0$ of equation (4)-(11), we solve equation (32)-(39) simultaneously to obtain

$$E_1 = \begin{pmatrix} S_h^{**} \\ E_h^{**} \\ I_h^{**} \\ H^{**} \\ R_h^{**} \\ S_n^{**} \\ E_n^{**} \\ I_n^{**} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{N_h^{**}M_1M_3}{\beta_hM_2} \\ \frac{\pi_hM_3}{\beta_hM_2}\left[\left[\frac{\beta_hM_2}{M_1M_3}-1\right]-\frac{\beta_{nh}\delta_{n0}}{\beta_n\delta_{h0}}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right]\right] \\ \frac{\pi_h}{\beta_h}\left[\left[\frac{\beta_hM_2}{M_1M_3}-1\right]-\frac{\beta_{nh}\delta_{n0}}{\beta_n\delta_{h0}}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right]\right] \\ \frac{\theta_1\pi_h}{M_4\beta_h}\left[\left[\frac{\beta_hM_2}{M_1M_3}-1\right]-\frac{\beta_{nh}\delta_{n0}}{\beta_n\delta_{h0}}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right]\right] \\ \left[\left[\frac{\beta_hM_2}{M_1M_3}-1\right]-\frac{\beta_{nh}\delta_{n0}}{\beta_n\delta_{h0}}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right]\right]\frac{1}{\delta_{h0}}\left[\frac{\alpha\rho\pi_hM_3}{\beta_hM_2}+\frac{\theta_1\theta_2\pi_h}{M_4\beta_h}\right] \\ \frac{\delta_{n0}(\gamma+\delta_{n0})N_n^{**}}{\beta_n\gamma} \\ \frac{\delta_{n0}\Pi_n}{\gamma\beta_n}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right] \\ \frac{\Pi_n}{\beta_n}\left[\frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}-1\right] \end{pmatrix} \quad (42)$$

Basic Reproduction Number(R_0)

The basic reproduction number(R_0) is the average number of secondary cases that arise as a result of an index case in an interactive community,(Diekmann, et al.,1990; Van Den Driessche & Watmough, 2002).

To compute the basic reproduction number(R_0) of the disease, we split our model in (4)-(11) into new infection terms as F and transition terms as V, represented by the following compartments as E_h(Exposed Human), I_h(Infected Human), E_n(Exposed), I_n(Infected Animal) and we employed the next generation matrix method, and subsequently obtain the largest eigenvalue of FV^{-1} to be the basic reproduction number of the model and hence $R_0 = \rho FV^{-1}$.

Applying partial derivative on F and V with respect to $E_h, I_h, E_n \& I_n$, yields the Jacobian matrix as

$$F = \begin{pmatrix} 0 & \frac{\beta_hS_h}{N_h} & 0 & \frac{\beta_{nh}S_h}{N_n} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta_nS_n}{N_n} \\ 0 & 0 & 0 & 0 \end{pmatrix}; \; V = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ -M_2 & M_3 & 0 & 0 \\ 0 & 0 & (\gamma+\delta_{n0}) & 0 \\ 0 & 0 & -\gamma & \delta_{n0} \end{pmatrix} \quad (43)$$

For DFE of F and taking of V in equation (43), becomes

$$F = \begin{pmatrix} 0 & \beta_h & 0 & \frac{\beta_{nh}\delta_{n0}\pi_h}{\pi_n\delta_{h0}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_n \\ 0 & 0 & 0 & 0 \end{pmatrix}; \; V^{-1} = \begin{pmatrix} \frac{1}{M_1} & 0 & 0 & 0 \\ \frac{M_2}{M_1M_3} & \frac{1}{M_3} & 0 & 0 \\ 0 & 0 & \frac{1}{(\gamma+\delta_{n0})} & 0 \\ 0 & 0 & \frac{\gamma}{\delta_{n0}(\gamma+\delta_{n0})} & \frac{1}{\delta_{n0}} \end{pmatrix} \quad (44)$$

For FV^{-1}, we obtain

$$FV^{-1} = \begin{pmatrix} \frac{\beta_hM_2}{M_3M_1} & \frac{\beta_h}{M_3} & \frac{\gamma\beta_{nh}}{\delta_{n0}(\gamma+\delta_{n0})} & \frac{\beta_{nh}}{\delta_{n0}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})} & \frac{\beta_n}{\delta_{n0}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (45)$$

For the eigenvalues of equation (45), we have

$$R_0 = \max(R_{0h}, R_{0n}) = \max\left\{\frac{\beta_hM_2}{M_3M_1}, \frac{\gamma\beta_n}{\delta_{n0}(\gamma+\delta_{n0})}\right\} = \max\left\{\frac{\beta_h\alpha(1-\rho)}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}, \frac{\gamma\beta_n}{(\gamma+\delta_{n0})\delta_{n0}}\right\} \quad (46)$$

Where $R_{0h} and \; R_{0n}$ in equation (46) are the basic reproduction number for both the human and animal population respectively.

Local Stability of the Disease-Free Equilibrium State

We compute the Jacobian of the model in (4)-(11), to establish the local stability of the disease free-equilibrium (DFE) state.

Theorem4: The DFE State of the Model in (4)-(11), is locally Asymptotically Stable, if $R_0 = \{R_{0h}, R_{0n}) < 1$, otherwise unstable.

Proof

We take the partial derivative of equation (14)-(21) with respect to each of $S_h, E_h, I_h, H, R_h, S_n, E_n, I_n$ at the DFE state in equation (41), i.e.

$$J(E_0) = \begin{pmatrix} -\delta_{h0} & 0 & -\beta_h & 0 & 0 & 0 & 0 & -\frac{\pi_h\beta_{nh}\delta_{n0}}{\pi_n\delta_{h0}} \\ 0 & -M_1 & \beta_h & 0 & 0 & 0 & 0 & \frac{\pi_h\beta_{nh}\delta_{n0}}{\pi_n\delta_{h0}} \\ 0 & M_2 & -M_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & -M_4 & 0 & 0 & 0 & 0 \\ 0 & \alpha\rho & 0 & \theta_2 & -\delta_{h0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_{n0} & 0 & -\beta_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\gamma+\delta_{n0}) & \beta_n \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -\delta_{n0} \end{pmatrix} \quad (47)$$

From equation (47), the eigenvalues of the matrix $J(E_0)$ are

$\nabla_1 = \nabla_2 = -\delta_{h0}, \nabla_3 = -\delta_{n0}, \; \nabla_4 = -M_4 = -(\theta_2 + \delta_{h0} + \delta_{h1})$

Which are negative and we expand the remaining matrix to give

$$\nabla^4 + a_1\nabla^3 + a_2\nabla^2 + a_3\nabla + a_4 = 0 \quad (48)$$

where

$a_1 = (M_1 + M_3 + (\gamma + \delta_{n0}) + \delta_{n0}), a_2 = (M_1M_3(1 - R_{0h}) + M_1(\gamma + \delta_{n0}) + M_1\delta_{n0} + M_3(\gamma + \delta_{n0}) + M_3\delta_{n0} + (\gamma + \delta_{n0})\delta_{n0}), a_3 = (M_1M_3((\gamma + \delta_{n0}) + \delta_{n0})(1 - R_{0h}) + (\gamma + \delta_{n0})\delta_{n0}(M_1 + M_3)), a_4 = (M_1M_3(\gamma + \delta_{n0})\delta_{n0}(1 - R_{0h}) + \beta_h\beta_r\gamma M_2)$

where $R_{0h} = \frac{\beta_hM_2}{M_1M_3}$

Following Routh-Hurwitz stability criterion in (Hurwitz,1964), all the eigenvalues of equation (47), have negative real part if $a_i > 0$, for $i = 1,2,3,4$ and $a_1a_2a_3 > a_3^2 + a_1^2a_4$.

Now,

$$a_1a_2a_3 - a_3^2 - a_1^2a_4 = a_3(a_1a_2 - a_3) - a_1^2a_4$$

And $a_i > 0, for \; i = 1,2,3,4$ since $R_{0h} < 1$, we show that $a_3(a_1a_2 - a_3) - a_1^2a_4 > 0$

$$\left([M_1{}^2M_3{}^2(1 - R_{0h})^2(M_1 + M_3)((\gamma + \delta_{n0}) + \delta_{n0}) + 2M_1M_3((\gamma + \delta_{n0}) + \delta_{n0})(1 - R_{0h})[M_1M_3((\gamma + \delta_{n0}) + \delta_{n0}) + (\gamma + \delta_{n0})\delta_{n0}(M_1 + M_3)] + M_1M_3((\gamma + \delta_{n0}) + \delta_{n0})(1 - R_{0h})[(\gamma + \delta_{n0})^2(M_1 + M_3 + \delta_{n0}) + \delta_{n0}{}^2(M_1 + M_3 + (\gamma + \delta_{n0}))] + 2M_1M_3((\gamma + \delta_{n0}) + \delta_{n0})[M_1M_3((\gamma + \delta_{n0}) + \delta_{n0}) + (\gamma + \delta_{n0})\delta_{n0}(M_1 + M_3)] + (\gamma + \delta_{n0})\delta_{n0}(M_1 + M_3)[(\gamma + \delta_{n0})^2(M_1 + M_3 + \delta_{n0}) + \delta_{n0}{}^2(M_1 + M_3 + (\gamma + \delta_{n0}))]) > ((M_1{}^2 + M_3{}^2 + (\gamma +$$

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$\delta_{n0})^2 + {\delta_{n0}}^2)[\, M_1M_3(\gamma+\delta_{n0})\delta_{n0}(1-R_{0h}) + \beta_h\beta_r\gamma M_2] + 2(M_1M_3 + M_1(\gamma+\delta_{n0}) + M_1\delta_{n0} + M_3(\gamma+\delta_{n0}) + M_3\delta_{n0} + (\gamma+\delta_{n0})\delta_{n0})[\, M_1M_3(\gamma+\delta_{n0})\delta_{n0}(1-R_{0h}) + \beta_h\beta_r\gamma M_2])$, since $R_{0h} < 1$

ThusE_0, of the model in (4)-(11) is locally asymptotically stable, since all eigenvalues have negative real part when$R_0 = (R_{0h}, R_{0n}) < 1$.

Global Stability of the Disease-Free Equilibrium State

To establish the disease is not dependent on the initial population size, we show that the disease free equilibrium state of the model in (4)-(11) is Globally Asymptotically Stable (GAS).

Lemma1: (Castillo-Chavez *et al.,* 2002), let the partition of equation (4)-(11), into

$$\frac{dX}{dt} = F(X,Y) and\ \frac{dY}{dt} = G(X,Y), G(X,Y) = 0 \quad (49)$$

Where $\in \mathbb{R}^m$, denote the population of the uninfected class and $Y \in \mathbb{R}^n$ denote the population of the infected class in equation (4)-(11) and $P(1)$ and $P(2)$ are assumed as

$P(1)$ $\frac{dX}{dt} = F(X,0), X^*$ is Globally Asymptotically Stable

$P(2)$ $G(X,Y) = AY - \hat{G}(X,Y), G(X,Y) \geq 0$, for $(X,Y) \in \phi$

Where $\frac{dG}{dY}, at\ X_0$ is a Metzler Matrix (m-matrix).

Note that if the equilibrium point$X_0 = (X^*, 0)$, is GAS whenever $R_0 = \{R_{0h}, R_{0n}) < 1$ and the conditions $P(1)$ and $P(2)$ are met.

Theorem5: The disease-free equilibrium state of equation (4)-(11) is globally asymptotically stable when $R_0 = \{R_{0h}, R_{0n}) < 1$

Proof

By Lemma1 and the assumption in $H(1)$ and $H(2)$

Let the uninfected class and infected class of equation (4)-(11) be $X = (S_h, H, R_h, S_n)^T$ and $Y = (E_h, , I_h, E_n, I_n)^T$ respectively, so that

$$\frac{dX}{dt} = \frac{d}{dt}\begin{pmatrix} S_h \\ H \\ R_h \\ S_n \end{pmatrix} = F = \begin{pmatrix} \Pi_h - \beta_h S_h \frac{I_h}{N_h} - \beta_{nh} S_h \frac{I_n}{N_n} - \delta_{h0} S_h \\ \theta_1 I_h - (\theta_2 + \delta_{h0} + \delta_{h1})H \\ \alpha\rho E_h + \theta_2 H - \delta_{h0} R_h \\ \Pi_n - \beta_n S_n \frac{I_n}{N_n} - \delta_{n0} S_n \end{pmatrix} \quad (50)$$

And

$$\frac{dY}{dt} = \frac{d}{dt}\begin{pmatrix} E_h \\ I_h \\ E_n \\ I_n \end{pmatrix} = G = \begin{pmatrix} \beta_h S_h \frac{I_h}{N_h} + \beta_{nh} S_h \frac{I_n}{N_n} - (\alpha + \delta_{h0})E_h \\ \alpha(1-\rho)E_h - (\theta_1 + \delta_{h0} + \delta_{h1})I_h \\ \beta_n S_n \frac{I_n}{N_n} - (\gamma + \delta_{n0})E_n \\ \gamma \mathrm{E}_n - \delta_{n0} I_n \end{pmatrix} \quad (51)$$

Since $E_h = I_h = E_n = I_n = 0$ (i.e. in the absence of the disease), then the solution of equation (51), gives

$$X^* = \left(\frac{\Pi_h}{\delta_{h0}}, 0, 0, \frac{\Pi_n}{\delta_{n0}}\right)$$

Hence, equation (52) is globally asymptotically stable for $\frac{dX}{dt} = F(X,0)$

Now, taking partial derivation of equation (51), we have A, but $G(X,Y) = AY - \hat{G}(X,Y), \Rightarrow \hat{G}(X,Y) = AY - G(X,Y)$, and we have

$$\hat{G}(X,Y) = \begin{pmatrix} -(\alpha+\delta_{h0}) & \beta_h \frac{S_h}{N_h} & 0 & \beta_{nh}\frac{S_h}{N_n} \\ \alpha(1-\rho) & -(\theta_1+\delta_{h0}+\delta_{h1}) & 0 & 0 \\ 0 & 0 & -(\gamma+\delta_{n0}) & \beta_n \frac{S_n}{N_n} \\ 0 & 0 & \gamma & -\delta_{n0} \end{pmatrix}\begin{pmatrix} E_h \\ I_h \\ E_n \\ I_n \end{pmatrix} - \begin{pmatrix} \beta_h S_h \frac{I_h}{N_h} + \beta_{nh} S_h \frac{I_n}{N_n} - (\alpha+\delta_{h0})E_h \\ \alpha(1-\rho)E_h - (\theta_1+\delta_{h0}+\delta_{h1})I_h \\ \beta_n S_n \frac{I_n}{N_n} - (\gamma+\delta_{n0})E_n \\ \gamma \mathrm{E}_n - \delta_{n0} I_n \end{pmatrix} = \begin{pmatrix} 0 & \beta_h I_h\left(1-\frac{S_h}{N_h}\right) & 0 & \beta_{nh} I_n\left(1-\frac{S_h}{N_n}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_n I_n\left(1-\frac{S_n}{N_n}\right) \\ 0 & 0 & 0 & 0 \end{pmatrix} \geq 0 \quad (53)$$

Since $\hat{G}(X,Y) \geq 0$ and A, is M-matrix(i.e. all element of the off-diagonal entries are nonnegative), then E_0 is globally asymptotically stable.

Local Stability of Endemic Equilibrium State

Theorem 6: The endemic equilibrium E_1 state of the model in equation (4)-(11) is locally Asymptotically Stable when $R_0 = \{R_{0h}, R_{0n}) > 1$

Proof

We generate the Jacobian Matrix of equation (4)-(11), at the endemic equilibrium point as

$$J(E_1) = \begin{pmatrix} -\beta_h \frac{I_h^{**}}{N_h^{**}} - \beta_{nh}\frac{I_n^{**}}{N_n^{**}} - \delta_{h0} & 0 & -\frac{\beta_h S_h^{**}}{N_h^{**}} & 0 & 0 & 0 & 0 & -\frac{\beta_{nh}S_h^{**}}{N_n^{**}} \\ \beta_h \frac{I_h^{**}}{N_h^{**}} + \beta_{nh}\frac{I_n^{**}}{N_n^{**}} & -M_1 & \frac{\beta_h S_h^{**}}{N_h^{**}} & 0 & 0 & 0 & 0 & \frac{\beta_{nh}S_h^{**}}{N_n^{**}} \\ 0 & M_2 & -M_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & -M_4 & 0 & 0 & 0 & 0 \\ 0 & \alpha\rho & 0 & \theta_2 & -\delta_{h0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\beta_n I_n^{**}}{N_n^{**}} - \delta_{n0} & 0 & -\frac{\beta_n S_n^{**}}{N_n^{**}} \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta_n I_n^{**}}{N_n^{**}} & -(\gamma+\delta_{n0}) & \frac{\beta_n S_n^{**}}{N_n^{**}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -\delta_{n0} \end{pmatrix} \quad (54)$$

Now the characteristic equation of equation (54) gives

$$\begin{vmatrix} b_0 - \delta_{h0} - \nabla & 0 & -\frac{\beta_h}{R_{oh}} & 0 & 0 & 0 & 0 & -\frac{\pi_h\beta_{nh}\delta_{n0}}{R_{oh}\pi_n\delta_{h0}} \\ -b_0 & -M_1 - \nabla & \frac{\beta_h}{R_{oh}} & 0 & 0 & 0 & 0 & \frac{\pi_h\beta_{nh}\delta_{n0}}{R_{oh}\pi_n\delta_{h0}} \\ 0 & M_2 & -M_3 - \nabla & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & -M_4 - \nabla & 0 & 0 & 0 & 0 \\ 0 & \alpha\rho & 0 & \theta_2 & -\delta_{h0} - \nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 - \delta_{n0} - \nabla & 0 & -\frac{\beta_n}{R_{on}} \\ 0 & 0 & 0 & 0 & 0 & -b_1 & -(\gamma+\delta_{n0}) - \nabla & -\frac{\beta_n}{R_{on}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -\delta_{n0} - \nabla \end{vmatrix} = 0 \quad (55)$$

Where $M_i\ , for\ i = 1-4$, is defined in equation (40) and $b_0 = -\beta_h \frac{I_h^{**}}{N_h^{**}} - \beta_{nh}\frac{I_n^{**}}{N_n^{**}} = -\delta_{h0}(R_{0h} - 1)\ and\ \ b_1 = -\frac{\beta_n I_n^{**}}{N_n^{**}} = -\delta_{n0}(R_{0n} - 1)$ (52)

The eigenvalues are $\nabla = -\delta_{h0}, \nabla = -M_4$ and remaining

characteristic equation in (55), gives

$$\nabla^6 + A_1\nabla^5 + A_2\nabla^4 + A_3\nabla^3 + A_4\nabla^2 + A_5\nabla + A_6 = 0 \quad (56)$$

Where

$$A_1 = M_1 + M_3 + \gamma + 2\delta_{n0} + \delta_{h0}R_{oh} + \delta_{n0}R_{on}$$

$$A_2 = M_1M_3 + \delta_{n0}(\gamma + \delta_{n0}) + (M_1{+}M_3)(\gamma + 2\delta_{n0}) + \delta_{h0}\delta_{n0}R_{oh}R_{on} + (\delta_{h0}R_{oh} + \delta_{n0}R_{on})[M_1 + M_3 + \gamma + 2\delta_{n0}] - M_2\frac{\beta_h}{R_{oh}}$$

$$A_3 = M_1M_3(\gamma + 2\delta_{n0}) + \delta_{n0}(\gamma + \delta_{n0})(M_1{+}M_3) + (M_1{+}M_3)(\gamma + 2\delta_{n0}) + \delta_{h0}\delta_{n0}R_{oh}R_{on}[M_1 + M_3 + \gamma + 2\delta_{n0}] + (\delta_{h0}R_{oh} + \delta_{n0}R_{on})[M_1M_3 + \delta_{n0}(\gamma + \delta_{n0}) + (M_1{+}M_3)(\gamma + 2\delta_{n0})] - M_2\frac{\beta_h}{R_{oh}}[\delta_{h0}R_{oh} + \delta_{n0}R_{on} + \gamma + 2\delta_{n0}] - M_2\frac{\beta_h}{R_{oh}}\delta_{h0}(R_{oh} - 1)$$

$$A_4 = M_1M_3\delta_{n0}(\gamma + \delta_{n0}) + \delta_{h0}\delta_{n0}R_{oh}R_{on}[M_1M_3 + \delta_{n0}(\gamma + \delta_{n0}) + (M_1{+}M_3)(\gamma + 2\delta_{n0})] + (\delta_{h0}R_{oh} + \delta_{n0}R_{on})[M_1M_3(\gamma + 2\delta_{n0}) + \delta_{n0}(\gamma + \delta_{n0})(M_1{+}M_3)] - M_2\frac{\beta_h}{R_{oh}}[\delta_{h0}R_{oh}\delta_{n0}R_{on} + (\gamma + 2\delta_{n0})(\delta_{h0}R_{oh} + \delta_{n0}R_{on}) + \delta_{n0}(\gamma + \delta_{n0})] - M_2\frac{\beta_h}{R_{oh}}\delta_{h0}(R_{oh} - 1)[\delta_{n0}R_{on} + (\gamma + 2\delta_{n0})]$$

$$A_5 = \delta_{h0}\delta_{n0}R_{oh}R_{on}[M_1M_3(\gamma + 2\delta_{n0}) + \delta_{n0}(\gamma + \delta_{n0})(M_1{+}M_3)] + M_1M_3\delta_{n0}(\gamma + \delta_{n0})[\delta_{h0}R_{oh} + \delta_{n0}R_{on}] + \gamma\frac{\beta_n}{R_{on}} - \gamma\frac{\beta_n}{R_{on}}\delta_{n0}(R_{on} - 1) - M_2\frac{\beta_h}{R_{oh}}\left[\delta_{h0}R_{oh}\delta_{n0}R_{on}(\gamma + 2\delta_{n0}) + \delta_{n0}(\gamma + \delta_{n0})(\delta_{h0}R_{oh} + \delta_{n0}R_{on}) + \gamma\frac{\beta_n}{R_{on}} + \gamma\frac{\beta_n}{R_{on}}\delta_{n0}(R_{on} - 1)\right] - M_2\frac{\beta_h}{R_{oh}}\delta_{h0}(R_{oh} - 1)[\delta_{n0}(R_{on} - 1)(\gamma + 2\delta_{n0}) + \delta_{n0}(\gamma + \delta_{n0})]$$

$$A_6 = \gamma\frac{\beta_n}{R_{on}}\delta_{h0}R_{oh}\delta_{n0}(R_{on} - 1) + \delta_{h0}R_{oh}\gamma\frac{\beta_n}{R_{on}} + \delta_{h0}\delta_{n0}R_{oh}R_{on}M_1M_3(\gamma + 2\delta_{n0}) - M_2\frac{\beta_h}{R_{oh}}\left[\delta_{h0}R_{oh}{\delta_{n0}}^2R_{on}(\gamma + \delta_{n0}) + \delta_{h0}\delta_{n0}R_{oh}(R_{on} - 1)\gamma\frac{\beta_n}{R_{on}} + \gamma\delta_{0h}R_{oh}\frac{\beta_n}{R_{on}}\right] - M_2\frac{\beta_h}{R_{oh}}\delta_{h0}(R_{oh} - 1)\left[\delta_{n0}R_{on}\delta_{n0}(\gamma + \delta_{n0}) + \gamma\frac{\beta_n}{R_{on}} + (R_{on} - 1)\delta_{n0}\gamma\frac{\beta_n}{R_{on}}\right]$$

From equation(56), we calculate the element of Routh array table gives

$$B_0 = \frac{A_1A_2 - A_3}{A_1}, B_1 = \frac{A_1A_4 - A_5}{A_1}, B_2 = \frac{A_1A_6 - 0}{A_1} = A_6, B_3 = \frac{B_0A_3 - B_1A_1}{B_0}, B_4 = \frac{B_0A_3 - B_2A_1}{B_0}, B_5 = \frac{B_1B_3 - B_0B_4}{B_3}, B_6 = \frac{B_2B_3 - 0}{B_3} = B_2, B_7 = \frac{B_4B_5 - B_3B_6}{B_5}$$

All the roots of the characteristic equation (55), have negative real, since the elements in the first column of the Routh array table are all are all positive and nonzero (Routh-Hurwitz, 1964).

Global Stability of Endemic Equilibrium State

Theorem 7: The endemic equilibrium E_1 state of the model in (4)-(11) is globally asymptotically stable (GAS) when $R_0 = \{R_{0h}, R_{0n}) > 1$

Proof

To prove the global stability of the endemic equilibrium stateE_1, we define the Lyapunov function as

$$V(t) = \left(S_h - S_h^{**} - S_h^{**}\ln\frac{S_h}{S_h^{**}}\right) + \left(E_h - E_h^{**} - E_h^{**}\ln\frac{E_h}{E_h^{**}}\right) + \left(I_h - I_h^{**} - I_h^{**}\ln\frac{I_h}{I_h^{**}}\right) + \left(H - H^{**} - H^{**}\ln\frac{H}{H^{**}}\right) + \left(S_n - S_n^{**} - S_n^{**}\ln\frac{S_n}{S_n^{**}}\right) + \left(E_n - E_n^{**} - E_n^{**}\ln\frac{E_n}{E_n^{**}}\right) + \left(I_n - I_n^{**} - I_n^{**}\ln\frac{I_n}{I_n^{**}}\right) \quad (58)$$

Differentiating equation (58) and transpose (4)-(11), gives

$$\frac{dV}{dt} = \left(1 - \frac{S_h^{**}}{S_h}\right)\left(\beta_h\frac{I_h^{**}}{N_h^{**}}S_h^{**} + \beta_{nh}\frac{I_n^{**}}{N_n^{**}}S_h^{**} + \delta_{h0}S_h^{**} - \beta_h\frac{I_h}{N_h}S_h - \beta_{nh}\frac{I_n}{N_n}S_h - \delta_{h0}S_h\right) + \left(1 - \frac{E_h^{**}}{E_h}\right)((\alpha + \delta_{h0})E_h^{**} - (\alpha + \delta_{h0})E_h) + \left(1 - \frac{I_h^{**}}{I_h}\right)((\theta_1 + \delta_{h0} + \delta_h)I_h^{**} - (\theta_1 + \delta_{h0} + \delta_{h1})I_h) + \left(1 - \frac{H^{**}}{H}\right)((\theta_2 + \delta_{h0} + \delta_{h1})H^{**} - (\theta_2 + \delta_{h0} + \delta_{h1})H) + \left(1 - \frac{S_n^{**}}{S_n}\right)\left(\beta_n\frac{I_n^{**}}{N_n^{**}}S_n^{**} + \delta_{n0}S_n^{**} - \beta_n\frac{I_n}{N_n}S_n - \delta_{n0}S_n\right) + \left(1 - \frac{E_n^{**}}{E_n}\right)((\gamma + \delta_{n0})E_n^{**} - (\gamma + \delta_{n0})E_n) + \left(1 - \frac{I_n^{**}}{I_n}\right)(\delta_{n0}I_n^{**} - \delta_{n0}I_n) \quad (59)$$

Simplifying equation (59) and applying the comparison principle that the arithmetic mean is greater or equal to the geometric mean (i.e. $\frac{1}{n}\sum_{i=1}^{n}x_i \geq \sqrt[n]{\prod_{i=1}^{n}x_i},\ i = 1,2,\dots,n$

Hence, from equation (59) we have

$$\left(1 - \frac{I_hS_hN_h^{**}}{I_h^{**}S_h^{**}N_h} - \frac{S_h^{**}}{S_h} + \frac{I_hN_h^{**}}{I_h^{**}N_h}\right) \leq 0, \left(1 - \frac{I_nS_hN_n^{**}}{I_n^{**}S_h^{**}N_n} - \frac{S_h^{**}}{S_h} + \frac{I_nN_n^{**}}{I_n^{**}N_n}\right) \leq 0, \left(2 - \frac{S_h^{**}}{S_h} - \frac{S_h}{S_h^{**}}\right) \leq 0, \left(2 - \frac{E_h}{E_h^{**}} - \frac{E_h^{**}}{E_h}\right) \leq 0, \left(2 - \frac{I_h}{I_h^{**}} - \frac{I_h^{**}}{I_h}\right) \leq 0, \left(2 - \frac{H}{H^{**}} - \frac{H^{**}}{H}\right) \leq 0, \left(1 - \frac{I_nS_nN_n^{**}}{S_n^{**}I_n^{**}N_n} - \frac{S_n^{**}}{S_n} + \frac{I_nN_n^{**}}{I_n^{**}N_n}\right) \leq 0, \left(2 - \frac{S_n}{S_n^{**}} - \frac{S_n^{**}}{S_n}\right) \leq 0, \left(2 - \frac{I_n}{I_n^{**}} - \frac{I_n^{**}}{I_n}\right) \leq 0, \left(2 - \frac{I_n}{I_n^{**}} - \frac{I_n^{**}}{I_n}\right) \leq 0$$

Therefore, by lyapunov theorem$\frac{dV}{dt} \leq 0$, for all $S_h, E_h, I_h, H, R_h, S_n, E_n, I_n > 0$ implying that E_1 is stable.

Now for asymptotic stability of E_1, we use Lasalle invariance principle i.e. $\frac{dV}{dt} = 0$, if and only if

$$\left(1 - \frac{I_hS_hN_h^{**}}{I_h^{**}S_h^{**}N_h} - \frac{S_h^{**}}{S_h} + \frac{I_hN_h^{**}}{I_h^{**}N_h}\right) = 0, \Rightarrow S_h = S_h^{**},\ I_h = I_h^{**}, I_n = I_n^{**}$$

Following similar process we conclude that, E_1 is asymptotically stable for $R_0 = \{R_{0h}, R_{0n}\} > 1$. E.g. (See Lasalle, 1976; El Hajji *et al*. 2015; El Hajji 2019a; Okolo *et al*. 2024).

Sensitivity Analysis of the Model

We performed sensitivity index on parameters of the basic reproduction number. These parameters includes contact rate β_h and β_n, the screening with therapy rate ρ, progression rate α, proportion rate $(1-\rho)$ due to latent period, isolation with treatment rate at θ_1 and progression to exposed class of the animal at a rate γ.

We used normalized forward sensitivity method defined as

$$X_V^{R_0} = \frac{\partial R_0}{\partial V}.\frac{V}{R_0} \quad (60)$$

Where V is the parameter rate and $R_0 = \{R_{0h}, R_{0n}\}$, is basic reproduction number for both population (Chitnis et al., 2008).

From equation (46), we have

$$R_0 = \{R_{0h}, R_{0n}\} = \left\{\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}, \frac{\gamma\beta_n}{(\gamma+\delta_{n0})\delta_{n0}}\right\}$$

Using equation (72), on equation (46), gives

$$X_{\beta_h}^{R_{0h}} = \frac{\partial}{\partial \beta_h}\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}.\frac{\beta_h}{R_{0h}} = 1 \quad (61)$$

$$X_{\rho}^{R_{0h}} = \frac{\partial}{\partial \rho}\left(\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}\right).\frac{\rho}{R_{0h}} = \frac{-\rho}{(1-\rho)} \quad (62)$$

$$X_{\alpha}^{R_{0h}} = \frac{\partial}{\partial \alpha}\left(\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}\right)\times\frac{\alpha}{R_{0h}} = \frac{\delta_{h0}}{(\alpha+\delta_{h0})} \quad (63)$$

$$X_{\theta_1}^{R_{0h}} = \frac{\partial}{\partial \theta_1}\left(\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}\right).\frac{\theta_1}{R_{0h}} = \frac{-\theta_1}{(\theta_1+\delta_{h0}+\delta_{h1})} \quad (64)$$

$$X_{\delta_{h0}}^{R_{0h}} = \frac{\partial}{\partial \delta_{h0}}\left(\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}\right).\frac{\delta_{h0}}{R_{0h}} = \frac{-\delta_{h0}[(\alpha+\delta_{h0})+(\theta_1+\delta_{h0}+\delta_{h1})]}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})} \quad (65)$$

$$X_{\delta_{h1}}^{R_{0h}} = \frac{\partial}{\partial \delta_{h0}}\left(\frac{\alpha(1-\rho)\beta_h}{(\alpha+\delta_{h0})(\theta_1+\delta_{h0}+\delta_{h1})}\right).\frac{\delta_{h1}}{R_{0h}} = \frac{-\delta_{h1}}{(\theta_1+\delta_{h0}+\delta_{h1})} \quad (66)$$

$$X_{\beta_n}^{R_{0n}} = X_{\beta_n}^{R_{0n}} = \frac{\partial}{\partial \beta_n}\left(\frac{\gamma\beta_n}{(\gamma+\delta_{n0})\delta_{n0}}\right)\times\frac{\beta_n}{R_{0n}} = 1 \quad (67)$$

$$X_{\gamma}^{R_{0n}} = \frac{\partial}{\partial \gamma}\left(\frac{\gamma(\gamma+\delta_{n0})^{-1}\beta_n}{\delta_{n0}}\right)\times\frac{\gamma}{R_{0n}} = \frac{\delta_{n0}}{(\gamma+\delta_{n0})} \quad (68)$$

$$X_{\delta_{n0}}^{R_{0n}} = \frac{\partial}{\partial \delta_{n0}}\left(\frac{\gamma\beta_n}{(\gamma+\delta_{n0})\delta_{n0}}\right)\times\frac{\delta_{n0}}{R_{0n}} = -\left(\frac{\gamma+2\delta_{n0})}{(\gamma+\delta_{n0})}\right) \quad (69)$$

DISCUSSION

We constructed and analyzed a mathematical model of Mpox infection, in the presence of early screening with therapy of the exposed and isolation with treatment of infected individuals.

The basic properties of the model's solution, such as existence and uniqueness, positivity and feasible region were shown.

The basic reproduction number of the disease was determined and the DFE state and EE state were obtained. The stability analysis results were shown in Theorems 4 and 5 to be locally and globally asymptotically stable at the DFE state when $R_0 < 1$, indicating infection will not create an epidemic and it was also shown at EE state to be locally and globally asymptotically stable when $R_0 > 1$, in Theorems 6 and 7, indicating infection will continue in the population and may lead to a pandemic.

The results of the sensitivity analysis of basic reproduction number with respect to the model parameters such as transmission rate, screening with therapy of exposed, progression from exposed class to infected class and isolation with treatment of infected individuals, negative indicates decrease in disease transmission with measures in place and positive indicates progression in transmission.

Conclusion

We constructed and analyzed a deterministic model of Mpox infection, for the management and control of the disease spread. We explore the impact of infection transmission pathways, early therapy on the exposed and Isolation of the infected class. Furthermore, the sensitivity index was performed. This analytical result has further motivated us to validate the work in this paper with numerical simulations using real data.

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