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Full Length Research Article

SOLUTION OF VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS USING CHEBYSHEV LEAST SQUARE METHOD

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ABSTRACT:
This paper investigates the application of the least squares method for obtaining numerical solutions to Volterra-Fredholm integro-differential equations. The least squares method is a well-established approach for solving integral equations, and in this study, it is utilized to find approximate solutions to such equations. To enhance the accuracy of the solutions, Chebyshev polynomials are used as basis functions for the approximation process. These polynomials are chosen due to their favorable convergence properties and their ability to provide accurate approximations over a wide range of problems. Several examples are included in this study to demonstrate the effectiveness and reliability of the proposed method. The numerical results obtained using the least squares method with Chebyshev polynomial approximations are compared with exact solutions, showing excellent agreement. The outcomes of this study indicate that the method is both efficient and reliable for solving Volterra-Fredholm integro-differential equations, offering a robust approach for practical applications.

Keywords: Volterra-fredholm integro-differential equations, Chebyshev polynomials, least square method.

INTRODUCTION

Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in two forms, namely

$$y^n(x) = g(x) + \lambda_1 \int_r^x k_1(x,t)y(t)dt + \lambda_2 \int_r^s k_2(x,t)y(t)dt \quad (1)$$

And

$$y^n(x,t) = g(x,t) + \lambda \int_0^t \int_\Omega^T G\big(x,t,\delta,\tau,y(\delta,\tau)\big)d\delta d\tau, (x,t) \in \Omega \times [0,T] \quad (2)$$

Where $g(x,t)$ and $G\big(x,t,\delta,\tau,u(\delta,\tau)\big)$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1,2,3$. It is interesting to note that (1) contains disjoint volterra and fredholm integral equations, whereas (2) contains mixed integrals. The unknown functions $y(x)$ and $y(x,t)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown function appears only inside the integral signs, the resulting equations are of first kind.

It is important to note that initial conditions should be given for volterra integro-differential equations to determine the particular solutions.

Over the last few years, numerous numerical methods have been presented to solve volterra-fredholm integro-differential equations. For example, integral collocation methods for the solution of higher-orders linear fredholm-volterra integro-differential equations was presented by Abubakar. A and Taiwo O.A (2014). (Akyuz *et al*., 2003) developed a matrix method to solve volterra-fredholm integral equations. Solution of volterra integral and integro-differential equations using modified Laplace Adomian decomposition method was presented by D. Rani and V. Mishra (2019). (Cardone, A. *et al*., 2018) Proposed the computational treatment of differential equations using collocation method for ordinary differential equation and presented its application with examples. Conclusion was drawn that the method presented yield desired result when compared with the exact solution. (Taiwo O.A *et al*., 2014) Employed multiple perturbed collocation Tau method to solve higher order linear and nonlinear boundary value problems with the aid of Chebyshev basis functions. (Caruntu. *et al*., 2021) Presented polynomial least square method for nonlinear fractional volterra and fredholm integro-differential equations. (Ahsan Fayez *et al*., 2018) Combined the least square method with Euler polynomials for finding the approximate solutions of integro-differential equations. Wazwaz (2010), Combined Laplace transform-Adomian decomposition method for handling nonlinear volterra integro-differential equations. In this study we presented Chebyshev least square method to solve Volterra Fredholm integro-differential equations. Other methods used by authors are He's Homotopy perturbation method (Abbasbany (2006, 2007). Other numerical techniques can be found in (Owolanke et al., 2021; Owolanke, Uwaheren, & Obarhua, 2017; Ogunbamike & Owolanke, 2022; Ogunbamike, Awolere, & Owolanke, 2021; Owolanke, Ogunbamike, & Adebayo, 2024; Yakusak & Owolanke, 2018; Owolanke & Ogunbamike, 2018).

Chebyshev and shifted Chebyshev polynomials

Chebyshev polynomials are sequence of orthogonal polynomials which are related to de-Moivre's formula and which can be defined recursively. One usually distinguishes between chebyshev polynomials of first kind which are denoted by T_n and chebyshev polynomials of second kind which are denoted by U_n.

Chebyshev polynomials of first kind

Chebyshev polynomials of first kind $T_p(x)$ is defined as:

$$T_p(x) = \cos(p\cos^{-1}x), \qquad -1 \le x \le 1 \quad (2)$$

Or equivalently

$$T_p(x) = \cos n\theta, \quad where \quad \theta = \cos^{-1}x \quad (3)$$

The few chebyshev polynomials of the first kind are;

n	$T_p(x)$
0	$T_0(x) = 1$
1	$T_1(x) = x$
2	$T_2(x) = 2x^2 - 1$
3	$T_3(x) = 4x^3 - 3x$
4	$T_4(x) = 8x^4 - 8x^2 + 1$
5	$T_5(x) = 16x^5 - 20x^3 + 5x$

The Shifted Chebyshev polynomials

For convenience and for the sake of problems that exist in intervals other than $-1 \le x \le 1, T_p(x)$ is in this subsection normalized to a general finite range $a \le x \le b$ as follows:

$$T^*_p(x) = \cos(pcos^{-1}x); \qquad -1 \le x \le 1 \qquad (4)$$

And the recurrence relation is given by

$$T^*_{p+1}(x) = 2xT^*_p(x) - T^*_{p-1}(x), \qquad p \ge 1$$

Where N is the degree of the polynomial.

In general, chebyshev polynomial valid in $a \le x \le b$ is given as

$$T^*_p(x) = \cos\left[Ncos^{-1}\left(\frac{2x-a-b}{a-b}\right)\right]; \qquad -1 \le x \le 1 \qquad (5)$$

And the recurrence relation is given as

$$T^*_{p+1}(x) = 2\left(\frac{2x-a-b}{a-b}\right)T^*_p(x) - T^*_{p-1}(x) \qquad (6)$$

Few terms of the shifted chebyshev polynomials valid in the interval [0, 1] are given below:

$$T^*_0(x) = 1$$
$$T^*_1(x) = 2x - 1$$
$$T^*_2(x) = 8x^2 - 8x + 1$$
$$T^*_3(x) = 32x^3 - 48x^2 + 18x - 1$$
$$T^*_4(x) = 128x^4 - 256x^3 + 100x^2 - 32x + 1$$
$$T^*_5(x) = 512x^5 - 128x^4 + 1120x^3 - 400x^2 + 50x - 1$$
$$T^*_6(x) = 204x^6 - 6144x^5 + 6912x^4 - 5484x^3 + 840x^2 - 72x + 1$$
$$T^*_7(x) = 32765x^8 - 131072x^7 + 212992x^6 - 40224x^5 + 84480x^4 - 21504x^3 + 26868x^2 - 128x + 1$$

METHODOLOGY

This section, we discussed the chebyshev least square method on the solution of volterra- fredholm integro-differential equations.

Consider the Volterra-Fredholm integro-differential equations of the form :

$$y^n(x) = g(x) + \lambda_1\int_0^x k_1(x,t)y(t)dt + \lambda_2\int_0^1 k_2(x,t)y(t)dt \qquad (3)$$

With the initial condition:

$$y_k(0) = \emptyset_k \qquad (4)$$

We assumed an approximate solution of the form:

$$y(x) = y_N(x) = \sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (5)$$

Where a_k , $k = 0(1)N$ are unknown constants to be determined and $T^*_k(x)$ is the shifted chebyshev polynomial basis function.

Differentiating equation (5) n-times to obtain:

$$y'_N(x) = \frac{d}{dx}\sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (6)$$

$$y''_N(x) = \frac{d^2}{dx^2}\sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (7)$$

$$y'''_N(x) = \frac{d^3}{dx^3}\sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (8)$$

$$y'^v_N(x) = \frac{d'^v}{dx'^v}\sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (9)$$

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$$y^n_N(x) = \frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*_k(x) \qquad (10)$$

Substituting the assumed approximate solution equations (5) and (10) into equation (3) to obtain:

$$\frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*_k(x) = g(x) + \lambda_1\int_0^x k_1(x,t)\sum_{k=0}^{N} a_k\, T^*_k(x)\,dt + \lambda_2\int_0^1 k_2(x,t)\sum_{k=0}^{N} a_k\, T^*_k(x)\,dt \qquad (11)$$

And therefore, the residual $R(u,x)$ will be:

$$R(u,x) = \frac{d^n}{dx^n}\sum_{k=0}^{N} a_k\, T^*_k(x) - g(x) - \lambda_1\int_0^x k_1(x,t)\sum_{k=0}^{N} a_k\, T^*_k(x)\,dt - \lambda_2\int_0^1 k_2(x,t)\sum_{k=0}^{N} a_k\, T^*_k(x)\,dt \qquad (12)$$

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Now minimizing the square of the residual error, i.e.

$$E(a_i \dots a_n) = \int_0^x \left[R\big(t, y(x)\big)\right]^2 dt \qquad x \in [0, t] \qquad (13)$$

From (13) the problem now is reduced to find the coefficient $a_i's$ which minimize E, is that $\frac{\partial E}{\partial a_i} = 0$ for each i=1, 2, …n. which will give a linear system of n- equation. The system of equations obtained is then solved to get values for the unknown constants. The values are now substituted in to the assumed solution given in equation (5) to get the approximate solution. It is important to note that when the problem contains some initial conditions, we first apply those conditions before implementing the least square method procedure to obtain the remaining number of required equations.

NUMERICAL EXAMPLES

Example 1: consider the third order volterra-fredholm integro-differential equation:

$$y'''(x) + y'(x) = g(x) + \int_0^x t y(t) dt + \int_0^1 y(t) dt \qquad (14)$$

Where $g(x) = \left(-\frac{x^5}{5} + \frac{x^4}{4} + 3x^2 - 2x + \frac{73}{12}\right)$, with the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -2 \qquad (15)$$

And the exact solution given by: $y(x) = x^3 - x^2$

Example 2: consider the second order volterra-fredholm integro-differential equation

$$y''(x) = g(x) - \int_0^x t y(t) dt - \int_0^1 y(t) dt \qquad (16)$$

Where $g(x) = \left(\frac{x^4}{4} + \frac{x^2}{2} + \frac{10}{3}\right)$, with the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \qquad (17)$$

And the exact solution given by: $y(x) = x^2 + 1$

Example 3: consider the second order volterra-fredholm integro-differential equation

$$y''(x) + y(x) = g(x) - \int_0^x t^2 y(t) dt - \int_0^1 y(t) dt \qquad (18)$$

Where $g(x) = \left(6x + x^3 - \frac{x^6}{6} - \frac{1}{4}\right)$, with the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \qquad (19)$$

And the exact solution given by: $y(x) = x^3$

Table 1: Numerical Results for Example 1 (CASE N=4 , N=6 and N=8)

x	Exact	Approx N=4	Error N=4	Approx. N=6	Error N=6	Approx. N=8	Error N=8
0.0	0.00000	0.00000	0.000e+00	0.00000	0.000e+00	0.00000	0.000e+00
0.1	-0.00900	-0.00900	0.000e+00	-0.00900	0.000e+00	-0.00900	0.000e+00
0.2	-0.03200	-0.03200	0.000e+00	-0.03200	0.000e+00	-0.03200	0.000e+00
0.3	-0.06300	-0.06300	0.000e+00	-0.06300	0.000e+00	-0.06300	0.000e+00
0.4	-0.09600	-0.09600	0.000e+00	-0.09600	0.000e+00	-0.09600	0.000e+00
0.5	-0.12500	-0.12500	0.000e+00	-0.12500	0.000e+00	-0.12500	0.000e+00
0.6	-0.14400	-0.14400	0.000e+00	-0.14400	0.000e+00	-0.14400	0.000e+00
0.7	-0.14700	-0.14700	0.000e+00	-0.14700	0.000e+00	-0.14700	0.000e+00
0.8	-0.12800	-0.12800	0.000e+00	-0.12800	0.000e+00	-0.12800	0.000e+00
0.9	-0.08100	-0.08100	0.000e+00	-0.08100	0.000e+00	-0.08100	0.000e+00
1.0	0.00000	0.00000	0.000e+00	0.00000	0.000e+00	0.00000	0.000e+00

Table 2: Numerical Results for Example 2 (CASE N=4 , N=6 and N=8)

x	Exact	Approx N=4	Error N=4	Approx N=6	Error N=6	Approx. N=8	Error N=8
0.0	1.00000	1.00000	0.000e+00	1.00000	0.000e+00	1.00000	0.000e+00
0.1	1.01000	1.01000	0.000e+00	1.01000	0.000e+00	1.01000	0.000e+00
0.2	1.04000	1.04000	0.000e+00	1.04000	0.000e+00	1.04000	0.000e+00
0.3	1.09000	1.09000	0.000e+00	1.09000	0.000e+00	1.09000	0.000e+00
0.4	1.16000	1.16000	0.000e+00	1.16000	0.000e+00	1.16000	0.000e+00
0.5	1.25000	1.25000	0.000e+00	1.25000	0.000e+00	1.25000	0.000e+00
0	1.36	1.36	0.00	1.360	0.000	1.3600	0.00

.6	000	000	0e+00	00	e+00	0	0e+00
0.7	1.49000	1.49000	0.000e+00	1.49000	0.000e+00	1.49000	0.000e+00
0.8	1.64000	1.64000	0.000e+00	1.64000	0.000e+00	1.64000	0.000e+00
0.9	1.81000	1.81000	0.000e+00	1.81000	0.000e+00	1.81000	0.000e+00
1.0	2.00000	2.00000	0.000e+00	2.00000	0.000e+00	2.00000	0.000e+00

Table 3: Numerical Results for Example 3 (CASE N=4 , N=6 and N=8)

x	Exact	Appro N=4	Error N=4	Approx N=6	Error N=6	Approx N=8	Error N=8
0.0	0.00000	0.00000	0.000e+00	0.00000	0.000e+00	0.00000	0.000e+00
0.1	0.00100	0.00100	0.000e+00	0.00100	0.000e+00	0.00100	0.000e+00
0.2	0.00800	0.00800	0.000e+00	0.00800	0.000e+00	0.00800	0.000e+00
0.3	0.02700	0.02700	0.000e+00	0.02700	0.000e+00	0.02700	0.000e+00
0.4	0.06400	0.06400	0.000e+00	0.06400	0.000e+00	0.06400	0.000e+00
0.5	0.12500	0.12500	0.000e+00	0.12500	0.000e+00	0.12500	0.000e+00
0.6	0.21600	0.21600	0.000e+00	0.21600	0.000e+00	0.21600	0.000e+00
0.7	0.34300	0.34300	0.000e+00	0.34300	0.000e+00	0.34300	0.000e+00
0.8	0.51200	0.51200	0.000e+00	0.51200	0.000e+00	0.51200	0.000e+00
0.9	0.72900	0.72900	0.000e+00	0.72900	0.000e+00	0.72900	0.000e+00
1.0	1.00000	1.00000	0.000e+00	1.00000	0.000e+00	1.00000	0.000e+00

DISCUSSION

Based on the findings from this work, we have successfully established that the shifted Chebyshev polynomials can as well be used as basis function in the formulation of least square method for solving Volterra-Fredholm integro-differential equations.

Conclusion

In this study, we presented the least square method to solve volterra-fredholhm integro-differential equations with the aid of shifted chebyshev polynomial basis functions as the approximate solution.

Three examples were considered with the method and we observed from the results obtained that the method is an accurate and reliable numerical scheme for solving volterra-fredholm integro-differential equation.

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