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# ENHANCING THE CONVERGENCE RATE OF A PRECONDITIONED ACCELERATED OVERRELAXATION METHOD FOR LARGE SPARSE LINEAR ALGEBRAIC SYSTEMS VIA A THIRD-LEVEL REFINEMENT STRATEGY

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## ABSTRACT

The accelerated overrelaxation (AOR) method is a widely used iterative method for solving large, sparse systems of linear equations due to its simplicity and low memory requirements. However, the AOR method may not always converge or may converge slowly for certain type of matrices. Preconditioning and refinement strategies are some of the techniques that have been introduced to overcome these limitations. This study presents the third refinement of a preconditioned accelerated overrelaxation iterative method, geared towards further enhancement of its convergence properties. It involves the application of a repeated refinement technique to a preconditioned AOR method to minimize its spectral radius, thereby reducing the iteration count and computational time. Theoretical convergence analysis and numerical experiments confirm the enhanced convergence properties, efficiency and accuracy of the refined method, significantly outperforming the existing AOR, its preconditioned variant and earlier refinements. This enhancement has far-reaching implications for solving large-scale linear systems in varied scientific and engineering application.

**Keywords:** Refinement, Accelerated Overrelaxation, Preconditioning, Iterative Method, Convergence

## INTRODUCTION

Consider the linear system

$$Bx = c \quad (1)$$

where the coefficient matrix $B$ is a nonsingular $n-$ square matrix with nonvanishing diagonal elements, $c$ is a column vector and $x$ is the vector of unknowns. A usual splitting of $B$ is obtained thus,

$$B = D - L_B - U_B \quad (2)$$

where $D, -L_B$ and $-U_B$ are the diagonal, strictly lower and strictly upper triangular parts of $A$ respectively. The system (1) can then be written as

$$D^{-1}(D - L_B - U_B)x = D^{-1}c \quad (3)$$

$$(I - L - U)x = b \quad (4)$$

$$Ax = b \quad (5)$$

where $A = D^{-1}B = I - L - U,\ b = D^{-1}c,\ L = D^{-1}L_B,\ U = D^{-1}U_B$.

The basic iterative methods for solving (1), or its equivalent form (5), include Jacobi, Gauss-Seidel, SOR and AOR methods. The Jacobi method is defined by

$$x^{(n+1)} = \mathcal{L}_J x^{(n)} + k_J, \qquad n = 0,1,2,\cdots \quad (6)$$

where $\mathcal{L}_J = D^{-1}(L_B + U_B)$ and $k_J = D^{-1}c$. Alternatively, $\mathcal{L}_J = L + U$, and $k_J = b$. Here, $\mathcal{L}_J$ is known as the Jacobian iteration matrix. Its spectral radius, $\rho(\mathcal{L}_J)$, is given by the relation

$$\rho(\mathcal{L}_J) = \bar{\mu} \quad (7)$$

where $\bar{\mu} = \max_i|\mu_i|$, $\mu_i\ (i = 1(1)n)$ is an eigenvalue of $\mathcal{L}_J$.

The Gauss-Seidel method has the iteration formula

$$x^{(n+1)} = \mathcal{L}_G x^{(n)} + k_G, \qquad n = 0,1,2,\cdots \quad (8)$$

where $\mathcal{L}_G = (D - L_B)^{-1}U_B$ and $k_G = (D - L_B)^{-1}c$. Or, $\mathcal{L}_G = (I - L)^{-1}U$, and $k_G = (I - L)^{-1}b$. The matrix $\mathcal{L}_G$ is known as the Gauss-Seidel iteration matrix, whose spectral radius is known to be the square of the spectral radius of the Jacobian iteration matrix, i.e.,

$$\rho(\mathcal{L}_G) = \bar{\mu}^2 \quad (9)$$

The iteration relation of the Successive Overrelaxation (SOR) method is described thus,

$$x^{(n+1)} = \mathcal{L}_\omega x^{(n)} + k_\omega \qquad n = 0,1,2,\cdots \quad (10)$$

for $\mathcal{L}_\omega = (D - \omega L_B)^{-1}\{(1-\omega)D + \omega U_B\}, k_\omega = (D - \omega L_B)^{-1}\omega c$. In conformity with (5), this can be written as $\mathcal{L}_\omega = (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}, k_\omega = (I - \omega L)^{-1}\omega b$. For the choice of relaxation parameter $\omega = 2/(1 + \sqrt{1 - \bar{\mu}^2})$, the spectral radius of the SOR iteration matrix is obtainable from

$$\rho(\mathcal{L}_\omega) = \left[1 - \sqrt{1 - \bar{\mu}^2}\right] / \left[1 + \sqrt{1 - \bar{\mu}^2}\right] \quad (11)$$

The Accelerated Overrelaxation (AOR) method introduced by Hadjidimos (1978) for the solution of (5) takes the form

$$x^{(n+1)} = \mathcal{L}_{r,\omega} x^{(n)} + k_{r,\omega} \qquad n = 0,1,2,\cdots \quad (12)$$

where $\mathcal{L}_{r,\omega}$ being the AOR iteration matrix is defined as $\mathcal{L}_{r,\omega} = (I - \omega L)^{-1}\{(1-r)I + (r-\omega)L + rU\}$, and $k_{r,\omega} = (I - \omega L)^{-1}rb$. Vatti and Mylapalli (2018) acknowledged the spectral radius of $\mathcal{L}_{r,\omega}$ as

$$\rho(\mathcal{L}_{r,\omega}) = \frac{\underline{\mu}\sqrt{\bar{\mu}^2 - \underline{\mu}^2}}{\sqrt{1 - \underline{\mu}^2}\left(1 + \sqrt{1 - \bar{\mu}^2}\right)} \quad (13)$$

under the restrictions $0 < \underline{\mu} < \bar{\mu}$ and $1 < \underline{\mu}^2 < \sqrt{1 - \bar{\mu}^2}$ for the choices $\omega = 2/(1 + \sqrt{1 - \bar{\mu}^2})$ and $r = [\left(1 - \underline{\mu}^2\right) - \sqrt{1 - \bar{\mu}^2}]/[\left(1 - \underline{\mu}^2\right)(1 + \sqrt{1 - \bar{\mu}^2})]$, where $\bar{\mu} = \max_i|\mu_i|$ and $\underline{\mu} = \min_i|\mu_i|$, $\mu_i\ (i = 1(1)n)$ is an eigenvalue of $\mathcal{L}_J$.

The rate of convergence of an iterative method such as the Jacobi, Gauss-Seidel, SOR or AOR methods is determined by the spectral radius of the associated iterative matrix. The goal in choosing an appropriate solver is to hand-pick a method whose associated matrix has a minimal spectral radius. However, the performance of these basic iterative methods is almost always constrained by the spectral radius of the associated iterative matrix, specifically when dealing with poorly conditioned or large systems. Hence, there is the need to devise ways and means of minimizing the spectral radius. An effective means of achieving just that is through preconditioning. The goal of preconditioning is to reduce the spectral radius of the associated iterative matrix so as to accelerate the convergence of the basic iterative methods. It involves the conversion of system (5) into the equivalent preconditioned system

$$PAx = Pb \qquad (14)$$

through the application of a nonsingular matrix $P$ called a preconditioner. Some notable preconditioning advances that exist in the literature include Ndanusa and Adeboye (2012), Miao *et al.* (2018), Wang (2019), Faruk and Ndanusa (2019), Song (2020), Ndanusa *et al.,* (2020) and Suleiman *et al.* (2024). Iterative refinement is another means of accelerating convergence of iterative methods for solving a linear system by performing iterations on the linear system whose right-hand side is the residual vector for successive approximations until satisfactory accuracy results are obtained. The refinement of AOR is obtained vide Vatti *et al.* (2018), wherein the refinement is achieved through the iteration

$$x^{(n+1)} = \mathcal{L}_{r,\omega}{}^2 x^{(n)} + d \qquad (15)$$

where $\mathcal{L}_{r,\omega}{}^2 = [(I-\omega L)^{-1}\{(1-r)I + (r-\omega)L + rU\}]^2$, $d = r[I + \mathcal{L}_{r,\omega}](I-\omega L)^{-1}b$. Similar researches along this direction include Kebede (2017), Eneyew *et al.* (2019), Eneyew *et al.* (2020)a, Eneyew *et al.* (2020)b, Teklehaymanot (2021) and Wangwa, *et al.* (2025).

## MATERIALS AND METHODS

### The Preconditioned Linear System

It is assumed that the coefficient matrix $B$ of (1) is an irreducible $L-\text{matrix}$ with weak diagonal dominance. Hadjidimos (1978) pointed out that If $B$ is an irreducible matrix with weak diagonal dominance, then it will be nonsingular with nonvanishing diagonal elements. A transformation matrix $P$ in the sense of Abdullahi and Ndanusa (2020) is then applied to system (5) resulting in the preconditioned system

$$\hat{A}x = \hat{b} \qquad (16)$$

where $\hat{A} = PA = (I + \hat{S})A, \hat{b} = Pb = (I+\hat{S})b$. Here, $I$ is the identity matrix and $\hat{S}$ has the structure defined as

$$\hat{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ -a_{31} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (17)$$

A usual splitting of the preconditioned coefficient matrix of system (16) into its diagonal ($\hat{D}$), strictly lower ($-\hat{L}$) and strictly upper ($-\hat{U}$) components is obtained thus

$$\hat{A} = \hat{D} - \hat{L} - \hat{U} \qquad (18)$$

System (18) has the equivalent form

$$\bar{A}x = \bar{b} \qquad (19)$$

with the splitting $\bar{A} = I - \bar{L} - \bar{U}$; where, $I$ is an identity matrix, $\bar{A} = \hat{D}^{-1}\hat{A}$ and $\bar{b} = \hat{D}^{-1}\hat{b}$.

### Third Refinement of Preconditioned AOR (TRPAOR) Method

Arising from Vatti *et al.* (2018), a refinement of the AOR method is obtained from (19),

$$(I-\bar{L}-\bar{U})x = \bar{b}$$
$$(I-\omega\bar{L})x + r(I-\bar{L}-\bar{U})x = (I-\omega\bar{L})x + r\bar{b}$$
$$(I-\omega\bar{L})x = (I-\omega\bar{L})x - r(I-\bar{L}-\bar{U})x + r\bar{b}$$
$$(I-\omega\bar{L})x = (I-\omega\bar{L})x + r(\bar{b} - \bar{A}x)$$
$$x = x + r(I-\omega\bar{L})^{-1}(\bar{b} - \bar{A}x)$$

Accordingly, the refinement of AOR is governed by the relation

$$x^{(n+1)} = x^{(n+1)} + r(I-\omega\bar{L})^{-1}(\bar{b} - \bar{A}x^{(n+1)}) \qquad (20)$$

The $x^{(n+1)}$ on the right-hand side of (20) is the $(n+1)th$ approximation of the AOR method applied to the system by (19). That is, $x^{(n+1)} = \mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b}\,, n = 0,1,2,\cdots$. Substituting this into (20) produces

$$x^{(n+1)} = \mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b} + r(I-\omega\bar{L})^{-1}[\{\bar{b} - (I-\bar{L}-\bar{U})\}(\mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b})]$$

$$x^{(n+1)} = \mathcal{L}_{r,\omega}x^{(n)} + 2r(I-\omega\bar{L})^{-1}\bar{b} - (I-\omega\bar{L})^{-1}[\{(rI - r\bar{L} - r\bar{U}) + (I - I + \omega\bar{L} - \omega\bar{L})\}(\mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b})]$$

$$= \mathcal{L}_{r,\omega}x^{(n)} + 2r(I-\omega\bar{L})^{-1}\bar{b} - (I-\omega\bar{L})^{-1}[(I-\omega\bar{L}) - \{(I-rI) + (r-\omega)\bar{L} + r\bar{U}\}(\mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b})]$$

$$= \mathcal{L}_{r,\omega}x^{(n)} + 2r(I-\omega\bar{L})^{-1}\bar{b} - [I - (I-\omega\bar{L})^{-1}\{(1-r)I + (r-\omega)\bar{L} + r\bar{U}\}][(\mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b})]$$

$$= \mathcal{L}_{r,\omega}x^{(n)} + 2r(I-\omega\bar{L})^{-1}\bar{b} - \mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b} + \mathcal{L}_{r,\omega}[(\mathcal{L}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b})]$$

$$= \mathcal{L}^2{}_{r,\omega}x^{(n)} + r(I-\omega\bar{L})^{-1}\bar{b} + \mathcal{L}_{r,\omega}r(I-\omega\bar{L})^{-1}\bar{b}$$

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}x^{(n)} + r(I + \mathcal{L}_{r,\omega})(I-\omega\bar{L})^{-1}\bar{b} \qquad (21)$$

Equation (21) defines the refinement of the AOR method for solving (19); where $\mathcal{L}^2{}_{r,\omega} = [(I-\omega\bar{L})^{-1}\{(1-r)I + (r-\omega)\bar{L} + r\bar{U}\}]^2$ is the iterative matrix of refinement of the AOR method. Its spectral radius is computed from the relation

$$\rho(\mathcal{L}^2{}_{r,\omega}) = \frac{\underline{\mu}^2(\bar{\mu}^2 - \underline{\mu}^2)}{(1-\underline{\mu}^2)\left(1+\sqrt{1-\bar{\mu}^2}\right)^2} \qquad (22)$$

Building upon this, and following Assefa and Teklehaymanot (2021), a refinement of AOR method for the solution of (9) is obtained thus,

$$(I-\bar{L}-\bar{U})x = \bar{b}$$
$$(I-\omega\bar{L})x + r(I-\bar{L}-\bar{U})x = (I-\omega\bar{L})x + r\bar{b}$$
$$(I-\omega\bar{L})x = (I-\omega\bar{L})x - r(I-\bar{L}-\bar{U})x + r\bar{b}$$
$$(I-\omega\bar{L})x = (I-\omega\bar{L})x + r(\bar{b} - \bar{A}x)$$
$$x = x + r(I-\omega\bar{L})^{-1}(\bar{b} - \bar{A}x)$$
$$x^{(n+1)} = x^{(n+1)} + r(I-\omega\bar{L})^{-1}(\bar{b} - \bar{A}x^{(n+1)}) \qquad (23)$$

Replacing $x^{(n+1)}$ appearing on the right-hand side of (23) by (21),

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}x^{(n)} + r(I+\mathcal{L}_{r,\omega})(I-\omega\bar{L})^{-1}\bar{b} + r(I-\omega\bar{L})^{-1}[\{\bar{b} - (I-\bar{L}-\bar{U})\}(\mathcal{L}^2{}_{r,\omega}x^{(n)} + r(I+\mathcal{L}_{r,\omega})(I-\omega\bar{L})^{-1}\bar{b})]$$

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$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}x^{(n)} + r\left[I + \mathcal{L}_{r,\omega}\right](I - \omega\bar{L})^{-1}\bar{b} + r(I - \omega\bar{L})^{-1}\bar{b} - r(I - \omega\bar{L})^{-1}(I - \bar{L} - \bar{U})\{\mathcal{L}^2{}_{r,\omega}x^{(n)} + r\left[I + \mathcal{L}_{r,\omega}\right](I - \omega\bar{L})^{-1}\bar{b}\}$$

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}x^{(n)} - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})\mathcal{L}^2{}_{r,\omega}x^{(n)} + r(I + \mathcal{L}_{r,\omega})(I - \omega\bar{L})^{-1}\bar{b} + r(I - \omega\bar{L})^{-1}\bar{b} - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})r(I + \mathcal{L}_{r,\omega})(I - \omega\bar{L})^{-1}\bar{b}$$

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}[I - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})]x^{(n)} + r[I + \mathcal{L}_{r,\omega} + I - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})(I + \mathcal{L}_{r,\omega})](I - \omega\bar{L})^{-1}\bar{b}$$

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}[I - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})]x^{(n)} + r[I + (I + \mathcal{L}_{r,\omega})(I - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U})(I + \mathcal{L}_{r,\omega}))](I - \omega\bar{L})^{-1}\bar{b}$$

Note that,

$$I - (I - \omega\bar{L})^{-1}(rI - r\bar{L} - r\bar{U}) = (I - \omega\bar{L})(I - \omega\bar{L})^{-1} - (I - \omega\bar{L})^{-1}rI + (I - \omega\bar{L})^{-1}r\bar{L} + (I - \omega\bar{L})^{-1}r\bar{U}$$
$$= (I - \omega\bar{L})^{-1}\{(1 - r)I + (r - \omega)\bar{L} + r\bar{U}\} = \mathcal{L}_{r,\omega}$$

$$x^{(n+1)} = \mathcal{L}^2{}_{r,\omega}\left[\mathcal{L}_{r,\omega}\right]x^{(n)} + r[I + (I + \mathcal{L}_{r,\omega})\mathcal{L}_{r,\omega}](I - \omega\bar{L})^{-1}\bar{b}$$

$$x^{(n+1)} = \mathcal{L}^3{}_{r,\omega}x^{(n)} + r\left[I + \mathcal{L}_{r,\omega} + \mathcal{L}^2{}_{r,\omega}\right](I - \omega\bar{L})^{-1}\bar{b} \qquad (24)$$

Equation (24) defines the second refinement of the AOR method for solving (19); where $\mathcal{L}^3{}_{r,\omega} = [(I - \omega\bar{L})^{-1}\{(1 - r)I + (r - \omega)\bar{L} + r\bar{U})\}]^3$ is the iterative matrix of the second refinement of the AOR method. Its spectral radius is computed from the relation

$$\rho(\mathcal{L}^3{}_{r,\omega}) = \left(\frac{\underline{\mu}\sqrt{\bar{\mu}^2 - \underline{\mu}^2}}{\sqrt{1 - \underline{\mu}^2}\left(1 + \sqrt{1 - \bar{\mu}^2}\right)}\right)^3 \qquad (25)$$

Following the same procedure established by Assefa and Teklehaymanot (2021), a third refinement of the AOR method for the solution of (19) is obtained as

$$x^{(n+1)} = \mathcal{L}^4{}_{r,\omega}x^{(n)} + r\left[I + \mathcal{L}_{r,\omega} + \mathcal{L}^2{}_{r,\omega} + \mathcal{L}^3{}_{r,\omega}\right](I - \omega\bar{L})^{-1}\bar{b} \qquad (26)$$

with $\mathcal{L}^4{}_{r,\omega} = [(I - \omega\bar{L})^{-1}\{(1 - r)I + (r - \omega)\bar{L} + r\bar{U})\}]^4$ being the iterative matrix of third refinement of AOR method, and its spectral radius computed as

$$\rho(\mathcal{L}^4{}_{r,\omega}) = \left(\frac{\underline{\mu}\sqrt{\bar{\mu}^2 - \underline{\mu}^2}}{\sqrt{1 - \underline{\mu}^2}\left(1 + \sqrt{1 - \bar{\mu}^2}\right)}\right)^4 \qquad (26)$$

## Numerical Example

Let the coefficient matrix $B$ of the linear system (1) be defined as

$$B = \begin{pmatrix} 107/112 & -1/56 & -337/1764 & -29/168 & -23/112 & -1/36 \\ -1/36 & 23/24 & -1/56 & -751/4032 & -19/112 & -23/112 \\ -1/42 & -13/72 & 107/112 & -1/48 & -25/126 & -29/168 \\ -1/48 & -3/14 & -611/3528 & 323/336 & -1/42 & -337/1764 \\ -1/36 & -31/168 & -5/24 & -97/576 & 23/24 & -1/56 \\ 0 & -11/72 & -1/6 & -3/16 & -11/72 & 41/42 \end{pmatrix}$$

## RESULTS AND DISCUSSION

The spectral radii of the iteration matrices of Jacobi, Gauss-Seidel. Successive Overrelaxation (SOR), Accelerated Overrelaxation (AOR), Preconditioned Accelerated Overrelaxation (PAOR), Refinement of Preconditioned Accelerated Overrelaxation (RPAOR), Second Refinement of Preconditioned Accelerated Overrelaxation (SRPAOR), and Third Refinement of Preconditioned Accelerated Overrelaxation (TRPAOR) methods are computed alongside their convergence rates employing the Maple 2019 mathematical software package. The results are presented in Tables 1 to 4. The following notations are adopted therein:

$\rho(\mathcal{L}_J) =$ spectral radius of Jacobi iteration matrix; $\rho(\mathcal{L}_G) =$ spectral radius of Gauss-Seidel (GS) iteration matrix; $\rho(\mathcal{L}_\omega) =$ spectral radius of SOR iteration matrix; $\rho(\mathcal{L}_{r,\omega}) =$ spectral radius of the AOR iteration matrix; $\rho(\bar{\mathcal{L}}_{r,\omega}) =$ spectral radius of Preconditioned AOR (PAOR) iteration matrix; $\rho(\overline{\mathcal{L}^2}_{r,\omega}) =$ spectral radius of Refinement Preconditioned AOR (RPAOR) iteration matrix; $\rho(\overline{\mathcal{L}^3}_{r,\omega}) =$ spectral radius of Second Refinement Preconditioned AOR (SRPAOR) iteration matrix; $\rho(\overline{\mathcal{L}^4}_{r,\omega}) =$ spectral radius of the Third Refinement Preconditioned AOR (TRPAOR) iteration matrix. And lastly, $R(\mathcal{L}_x)$ refers to the rate of convergence of the iteration method whose iteration matrix is denoted by $\mathcal{L}_x$.

***Table 1*** *Eigenvalues, parameters and spectral radii of PAOR, RPAOR, SRPAOR and TRPAOR iteration matrices*

| **Eigenvalues** | | **Parameters** | | **Spectral radii** | | | |
|---|---|---|---|---|---|---|---|
| $\underline{\mu}$ | $\bar{\mu}$ | $r$ | $\omega$ | $\rho(\bar{\mathcal{L}}_{r,\omega})$ | $\rho(\overline{\mathcal{L}^2}_{r,\omega})$ | $\rho(\overline{\mathcal{L}^3}_{r,\omega})$ | $\rho(\overline{\mathcal{L}^4}_{r,\omega})$ |
| 0.002713 | 0.627936 | 0.124688 | 1.124691 | 0.000958 | $9.182 \times 10^{-}$ | $8.798 \times 10^{-}$ | $8.431 \times 10^{-}$ |

***Table 2*** *Eigenvalues, parameters and spectral radii of Jacobi, GS, SOR and AOR iteration matrices*

| **Eigenvalues** | | **Parameters** | | **Spectral radii** | | | |
|---|---|---|---|---|---|---|---|
| $\underline{\mu}$ | $\bar{\mu}$ | $r$ | $\omega$ | $\rho(\mathcal{L}_J)$ | $\rho(\mathcal{L}_G)$ | $\rho(\mathcal{L}_\omega)$ | $\rho(\mathcal{L}_{r,\omega})$ |
| 0.025476 | 0.642862 | 0.1322 | 1.1325 | 0.642862 | 0.413271 | 0.132514 | 0.009269 |

***Table 3*** *Convergence rate of PAOR, RPAOR, SRPAOR and TRPAOR methods*

| $R(\bar{\mathcal{L}}_{r,\omega})$ | $R(\overline{\mathcal{L}^2}_{r,\omega})$ | $R(\overline{\mathcal{L}^3}_{r,\omega})$ | $R(\overline{\mathcal{L}^4}_{r,\omega})$ |
|---|---|---|---|
| 6.950420 | 13.900840 | 20.851260 | 27.801680 |

***Table 4*** *Convergence rate of Jacobi, GS, SOR and AOR methods*

| $R(\mathcal{L}_J)$ | $R(\mathcal{L}_G)$ | $R(\mathcal{L}_\omega)$ | $R(\mathcal{L}_{r,\omega})$ |
|---|---|---|---|
| 0.441824 | 0.883644 | 2.021063 | 4.680987 |

Table 1 reveals that for the choices $r = 0.124688$, $\omega = 1.124691$, the third refinement of preconditioned AOR (TRPAOR) exhibits faster convergence than the SRPAOR, RPAOR and PAOR methods, i.e., $\rho(\bar{\mathcal{L}}_{r,\omega}) < \rho(\overline{\mathcal{L}^2}_{r,\omega}) < \rho(\overline{\mathcal{L}^3}_{r,\omega}) < \rho(\overline{\mathcal{L}^4}_{r,\omega}) < 1$. In Table 2, for the choices $r = 0.132232$, $\omega = 1.132514$, the classical AOR, as expected, converges faster than the SOR, GS and Jacobi methods in that order, i.e., $\rho(\mathcal{L}_J) < \rho(\mathcal{L}_G) < \rho(\mathcal{L}_\omega) < \rho(\mathcal{L}_{r,\omega}) < 1$. Table 3 and Table 4 compares the rates of convergence for various methods. That is, $R(\mathcal{L}_J) < R(\mathcal{L}_G) < R(\mathcal{L}_\omega) < R(\mathcal{L}_{r,\omega}) < R(\bar{\mathcal{L}}_{r,\omega}) < R(\overline{\mathcal{L}^2}_{r,\omega}) < R(\overline{\mathcal{L}^3}_{r,\omega}) < R(\overline{\mathcal{L}^4}_{r,\omega})$.

**Conclusion**

The main goal of iteration methods for linear systems is to devise means and ways of reducing the spectral radius of the corresponding iteration matrix, which must be shown to be less than 1, that is, $\rho(\mathcal{L}) < 1$, where $\rho(\mathcal{L}) = \max_i |\lambda_i|, \lambda_i \ (i = 1(1)n)$ is an eigenvalue of the iteration matrix $\mathcal{L}$ of the corresponding iteration method; and the smaller it is, the faster the method's rate of convergence. A combination of preconditioning and refinement techniques has been adopted to improve the convergence rate of the AOR method for solving linear systems with an irreducibly diagonally dominant $L$ −matrix coefficient matrix. It was established that the third refinement of preconditioned AOR converges approximately 23 times faster than the classical AOR method.

$\max_i |\lambda_i|, \lambda_i \ (i = 1(1)n)$ is an eigenvalue of $\mathcal{L}$

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