MATHEMATICAL MODELS FOR VEHICLE DEPRECIATION UNDER BOUNDED ROUTINE MAINTENANCE AND REPLACEMENT USING FINITE DIFFERENCE METHODS

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ABSTRACT

In this paper, we seek to improve an earlier replacement model that uses dynamic programming to analyze the impact of depreciation on state-owned vehicles. A common assumption associated with the model is that all return functions and decision possibilities are uniform or identical. The method of finite differences is applied to determine the optimal replacement time for automobiles whose economic value deteriorates over time, thereby increasing maintenance and operating costs. The objective of the model is to minimize the average yearly cost of operating vehicles whose costs of maintenance increase with time, while the scrap value is constant. It is assumed that the time value of money is negligible with a zero-interest rate. Hence, the calculation is based on the annual average cost. It is observed from the numerical illustration of the model that the optimal life of a vehicle can be determined by comparing the increased running cost to the decreased depreciation value. The two models presented in this article have a computational advantage over existing ones in terms of assisting policy makers to determine vehicles' optimal replacement policy. The policies include deciding which vehicle should be repaired or replaced, thereby averting losses of human and material resources due to accidents.

Keywords: Depreciation, vehicles, replacement, maintenance

INTRODUCTION

Maintenance is a recurring and routine activity carried out by industrial organizations on production equipment or facilities to keep them in good working condition for optimal performance. It is a common tradition in many industrial organizations to purchase vehicles and machines, use them to work for sometimes after which they are either service, repair, or discard them when they stop working. In Swathi (2020), vehicles' maintenance is classified into two categories: Breakdown maintenance and planned maintenance. Under breakdown maintenance, repair or maintenance is carried out only when the vehicles breakdown or fails to work, while in planned maintenance, repair and maintenance are planned to prevent sudden or random failure, as reported in Bethuyne (1998) and Penabad et al. (2018). Replacement models have been classified into two groups in Abdelwali et al. (2024):

(a) Replacement of vehicles and machines that gradually deteriorate in the course of time as a result of wear and tear of their components. These models aim at assessing the economic life of the machine as to whether it should be replaced or repaired.

(b) Probabilistic models which assess the probability of vehicles and machines that suddenly fail by considering the operating costs, maintenance cost and their revenue generation.

Scarf (2025) states that maintenance modeling can assist in the design and building of maintenance management information systems for operating management policy. Machine replacement strategy can be applied to vehicles, planes, refinery equipment, etc. According to Afshar-Nadjafi and Afshar-Nadjafi (2017), during inspection, machines status is classified into three states: partial failure, combined failure, and complete failure. According to Gautam and Shrivastava (2017) and Borowski and Lubomir (2023), the challenges encountered in the maintenance of vehicles/equipment are attributed to the following factors:

(a) The advent of modern technology, which makes the current vehicles/equipment obsolete, is being experienced in Information and Communication Technology (ICT) industries.

(b) Advent of rapid repair and diagnosis tools.

(c) Effective training and utilization of maintenance staff.

(d) Local production of genuine spare parts to replace the imported ones.

(e) Effective maintenance of old vehicles and machines to last longer.

The problems caused by the lack of maintenance of vehicles and equipment are undeniable. These problems include shortages of essential products such as petroleum products due to breakdown of refinery equipment, losses of human and material resources as a result of bad roads. Many private and public educational institutions and industries have abandoned vehicles wasting away on their premises. These vehicles include official cars, staff, and students' buses meant to convey them to and from school. Some of them need minor or major repair /maintenance, while others need complete replacement. The irony is that the more these vehicles are unattended to, the more they depreciate or deteriorate. This scenario is also applicable to equipment (such as computers, generators) and facilities such as buildings, roads, railways, and airports, some of which have become death traps to innocent citizens. There have been reports of collapsed buildings in many cities due to a lack of maintenance or repair of which entire families were wiped out. Many incidents of air crashes or road accidents where a lot of people died have been attributed to either bad roads, brakes, or engine failures, as a result of which both passengers and drivers/pilots died. Thus, lack of maintenance of equipment and facilities has led to loss of properties worth billions of naira and countless numbers of people. For example, the 1992 air crash involving Nigeria Air Force C-130 plane killed 200

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passengers on board, according to Punch Newspaper, 1 August 2023. Hence, there is a need to develop mathematical models such as the ones presented in this article to address these ugly trends of lack of maintenance.

Mardi and Arai (2012) and Abdelwali et al. (2024) opined that vehicles/machine optimal replacement policy can be determined if there is reliable data on the revenue it generates and the cost of its maintenance over a given period of time. In a replacement model presented in Hritonenko and Yatsenko (2008), on the dynamics of asset lifetime under technological changes, it is observed that vehicles that diminish in efficiency are usually associated with high maintenance and operating costs, while the ones that do not diminish in efficiency perform optimally for a period of time, then suddenly fail. A mathematical model that uses an efficiency and dependability approach to determine vehicles' optimal replacement policy is studied in Galar et al. (2012) and Gabdullin and Makhmutov (2021).

Military and industrial equipment performance decreases in efficiency as a result of constant usage over a long period of time. Some of the equipment breakdown or wears out and may need maintenance or repair. For example, vehicles used for transportation, aircraft, production machines, etc., need regular maintenance, which incurs operating costs, Swathi (2020). After a long period of time, some of the equipment becomes too expensive to maintain or repair due to the high cost of spare parts, hence they are usually replaced with new ones. Vehicles/Machines replacement could be expanded to include items that do not deteriorate, but with time suddenly fail. Examples of such items are electric bulbs. The challenge for managers is how to find which of the items are to be replaced and when.

The objective of replacement models, as stated in Khaliulin et al. (2021), is to determine the costs associated with the failed item, the cost of replacing the item, and the sum of the cost of the item. Another scenario that could necessitate the replacement of machines, as reported in Sultanov and Makhmutov (2020), is obsolescence due to the advent of new models. The reason for replacement in this case is driven by the emergence of modern equipment with new technology for better performance in terms of efficiency.

The costs of repair and maintenance of vehicles vary with time. Hence, replacement of vehicles or equipment becomes necessary if the cost of maintenance is considered to be higher than the cost of purchasing a new one. In making decisions about replacing vehicles/equipment, the age or choice of the vehicle and equipment is usually considered (Swathi, 2020; Abdelwali et al., 2024). The desire to meet high demand for goods and services in the global market forces manufacturers to ensure their equipment is in good condition to perform optimally. Manufacturers at times replace the old equipment with modern ones that have higher performance, productivity, and profitability (Galar et al., 2012). The total adjusted cost value of a vehicle has been defined in Penabad et al. (2018) as the vehicle's purchase cost + operating cost + maintenance cost + vehicle resale price. The key performance measure of a vehicle or piece of equipment hinges on the cost of its maintenance or replacement during its lifetime. Lei et al. (2023) remarked that the efficiency and effectiveness of vehicles/equipment decrease with age, resulting in a decline in productivity and profitability.

A discount rate is a term used to describe the cost incurred at different times or the time value of money. Several models in the literature aim at either estimating the equivalent present value of earlier costs or the equivalent value of the future costs of vehicles. Mathematical models that study the optimum replacement time of vehicles/machines abound in the literature. For example optimal lifetime of a machine, which uses nonlinear equations and indefinite integral to study optimum replacement time of machines, is reported in Boucekkine et al. (1997), and Hritonenko and Yatsenko (2005). Dynamic programming models that consider machine replacement lifetime with finite and infinite horizons of technological advancement are developed in Bethunye (1998), Hritonenko and Yatsenko (2008), and Mardi and Arai (2012). Hritonenko and Yatsenko (2007) use an integral model to evaluate the economic lifetime of machines with respect to technological changes and concluded that the economic lifetime of machines becomes shorter under intensive usage.

An optimal replacement model which minimizes the cost of maintenance of old vehicles/machines for capacity building is studied in Karri (2007), while a discounted cost parameter model to solve vehicles replacement problem over a specified period of time is discussed in Lei et al. (2023). Aldaihami (2017) used a fuzzy multi-objective integer programming technique to model the vehicle replacement problem. Gautam and Shrivastava (2017) developed a model that studies vehicles/machines with repeated failure and deterioration by using periodic inspection in order to know whether to repair or replace them. Afenyo et al. (2021) stated that the problem of vehicles' or machines' maintenance is more severe in developing countries than in developed ones because of differences in their level of technological advancement. Mathematical formulas such as linear fitting, exponential, and polynomial equations are used to predict optimum maintenance and repair costs of faulty vehicles or machines as reported in Sakhapov et al. (2018), Vdovin and Stronganov (2020), and Khaliulin et al. (2021). Fixed cost and operational cost constitute the major proportion of the total cost of production. A machine operational cost consists of the cost of maintenance and repair, while machinery ownership costs include interest on investment, insurance, taxes, and housing facility, as stated in Sakhapov et al. (2018) and Sultanov and Makhmutov (2020). A model that analyzes the impact of depreciation on state transportation vehicles and equipment using dynamic programming is developed in Lei et al. (2023).

Many mathematical replacement models based on decision-making, such as the ones earlier cited in this paper, deal with an unending history of actions known as an unbounded horizon. In these models, earlier decisions affect the present decisions while the current decisions influence future decisions. On the other hand, some models select specific terminal conditions (also known as bounded horizon), such as a minimum productivity level or workforce, in order to account for the future state of the system being modeled. A common assumption associated with the above-described scenarios is that all return functions, demand requirements, and decision possibilities are uniform or identical. (See Gupta and Hira (2005) and Lei et al. (2023)).

In this paper, we seek to expand the Lei et al. (2023) replacement model, which uses a dynamic programming technique to analyze

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the impact of depreciation of state-owned vehicles by applying the method of finite difference to find the optimum replacement time of automobile vehicles whose economic value deteriorates with time, thereby causing an increase in maintenance and operational costs.

MATERIALS AND METHODS
Mathematical Symbols
K = Purchase cost of the vehicle.
H = Value of the scrap of the vehicle.
N = Number of years the vehicle will be used.
Z_{ave} = Yearly average total cost of the vehicle.
$Z_{(n)}$ = Costs of operating and maintaining the vehicle at time t.
$\int_0^n z(t)\,dt =$
the total maintenance cost incurred in the period of n − years

Model 1: Vehicle with increasing Repair and Maintenance costs
Case 1: For Continuous Variable time t
The objective of the model is to determine the value of n that will minimize Z(n) total costs of operating and maintaining the vehicle during the period of n years. Hence, the yearly cost of the vehicle at time t = purchase cost – scrap value + maintenance cost at time t. Hence, the total cost incurred during n years.

$$Z(n) = K - H + \int_0^n z(t)\,dt \qquad (1)$$

While the average yearly cost incurred on the vehicle is

$$Z_{ave} = \frac{1}{n}\left[K - H + \int_0^n z(t)\,dt \right] \qquad (2)$$

The minimum value of Z_{ave} for n years is obtained by differentiating Equation (2) with respect to n. That is

$$\frac{d}{dn}\left(Z_{ave}\right) = \frac{1}{n^2}\left(K - H\right) - \frac{1}{n^2}\int_0^n z(t)\,dt + \frac{1}{n}Z(n)$$
$$(3)$$

If we set $\dfrac{d}{dn}\left(Z_{ave}\right) = 0,$ in Equation (3) we get

$$Z(n) = \frac{1}{n}\left[K - H + \int_0^n z(t)\,dt = Z_{ave} \right. \qquad (4)$$

Hence, the vehicle should be replaced when the yearly average cost to date equals the current cost of maintenance. (See Gupta and Hira (2005) and Lei et al. (2023)).

Case 2: For Discrete Variable time t.
In the case of a discrete variable time t, we simply replace the integral in the right-hand side of Equation (1) with a summation sign. That is, the total cost incurred in n-period years is:

$$Z(n) = K - H + \sum_{i=0}^{n} Z(t) \qquad (5)$$

∴ The yearly average cost incurred on the vehicle is

$$Z(n) = \frac{1}{n}\left[K - H + \sum_{i=0}^{n} Z(t) \right. \qquad (6)$$

Hence, n is optimal at the smallest average yearly cost using Equation (6)

Model 2: Replacement of vehicles whose maintenance costs increase with time, and the value of money varies.
Model Description: In this model, the value of money is assumed to vary with time. Hence, the present value of the money to be spent in years to come must be considered in the mathematical formulation of the model. This could be in the form of interest rate, inflation rate, or both. Let j be the inflation rate or the sum of interest and inflation rate annually. Hence, a naira used for purchase in the present year will be equivalent to (I + j) in a year, (I +j)² for two years, and (I +j)ⁿ in n years period. Therefore, the payment to be made for a naira after n years equals paying (I +j)⁻ⁿ in the current year. The value (I +j)⁻ⁿ is referred to as the current worth of one naira spent n years from now. This could be expressed as the present worth of a naira spent in n years. That is

$$(1 + j)^{-n} = U^n \qquad (7)$$
Where
$$(1 + j)^{-1} = \frac{1}{1 + j} \qquad (8)$$

Equation (8) is called the rate of discount, $(u < 1)$.

In order to calculate the optimal replacement policy of a vehicle, we have to consider the purchase price and the running costs. Let K be the purchase price used for the vehicle and $Q_1, Q_2,... Q_n$ be the costs of running the vehicle respectively. If we assume that the vehicle scrap value is zero and that at the beginning of each year, payments are made. Then the current worth of the expenditure on the vehicle in n-years can be expressed as

$$P_n = K + Q_1 + UQ_2 + U^2 Q_3 + ... + u^{n-1}Q_n \qquad (9)$$

Therefore, P_n is the current amount of money needed to pay for all future costs associated with acquiring and operating the vehicle if it is to be replaced after n years. Our interest is to find an interval of replacement function such that when P_n value increases in proportion to an increase in n-years, the replacement of the vehicle after (n+1) years is greater than when it is replaced after n years in terms of worth. This can be achieved if we assume P_n to be the amount the driver invested by borrowing money at j interest rate, which is to be paid back yearly in fixed payments throughout the life of the vehicle. Thus, by the end of n-years, the driver would have paid off the total cost of the vehicle. By applying Equation (9), the current value of fixed yearly payments of each value y, for n-years can be expressed as:

$$y + uy = u^2 y + ... + u^{n-1} y = \frac{1 - U^n}{1 - U} y \qquad (10)$$

Since this is equivalent to the P_n sum borrowed, we have

$$P_n = \frac{1 - U^n}{1 - U} y \qquad (11)$$
Where
$$y = \frac{1 - U^n}{1 - U} P_n \qquad (12)$$

Hence, the best interval period of replacing the vehicle is the one that will minimize the value y stated in Equation (12). Since $1 - U$ is a positive constant, the period at which the vehicle should be replaced is that of n, which will minimize the function:

$$E_n = \frac{P_n}{1 - U^n} \qquad (13)$$

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The method of finite differences can now be used to determine the minimum value of E_n since n value must be discrete (Gupta and Hira, 2005).

Thus, E_n will have a minimum value if

$$\Delta E_{n-1} < 0 < \Delta E_n \qquad (14)$$

And $\Delta E_n = E_{n+1} - E_n \qquad (15)$

By substituting Equation (13) into (15) we have

$$\Delta E_n = \frac{P_{n+1}}{1-U^{n+1}} - \frac{P_n}{1-U^n} \qquad (16)$$

$$= \frac{\left(1-U^n\right)P_{n+1} - (1-U^{n+1})P_n}{\left(1-U^{n+1}\right)\left(1-U^n\right)}$$

$$\Delta E_n = \frac{1}{\left(1-U^{n+1}\right)\left(1-U^n\right)}\left[\left(P_{n+1}-P_n\right)+(U^{n+1}P_n-U^nP_{n+1})\right] \qquad (17)$$

By applying the method of finite differences to Equation (9), we have

$$P_{n+1} = (K+Q_1+UQ_2+...+U^{n-1}Q)+U^nQ_{n+1} = P_n+UQ_{n+1}$$

Thus, from Equation (17), we get Equation (18) as follows:

$$\Delta E_n = \frac{1}{\left(1-U^{n+1}\right)\left(1-U^n\right)}\left[\left(U^nQ_{n+1}\right)+U^{n+1}P_n-U^n\left\{P_n+U^nQ_{n+1}\right\}\right]$$

$$= \frac{1}{\left(1-U^{n+1}\right)\left(1-U^n\right)}\left[U^nQ_{n+1}(1-U^n)-U^nP_n(1-U)\right]$$

$$= \frac{U^n\left(1-U\right)}{\left(1-U^{n+1}\right)\left(1-U^n\right)}\left[\frac{1-U^n}{1-U}Q_{n+1}-Pn\right] \qquad (18)$$

Where

$$\left[\frac{1-U^n}{1-U}Q_{n+1}-Pn\right]$$

is a positive constant.

Therefore, E_n always has the same value as the one in the brackets. Thus, from Equation (14), n will have an optimal value if

$$\left[\frac{1-U^{n-1}}{1-U}Q_n-P_{n-1}<0<\frac{1-U^n}{1-U}Q_{n+1}-P_n\right] \qquad (19)$$

From the right-hand side of Equation (19), we have

$$\frac{1-U^n}{1-U}Q_{n+1}-P_n > 0$$

or

$$Q_{n+1} > Pn.\frac{1-U}{1-U^n}$$

Or

$$Q_{n+1} > Pn/\frac{1-U^n}{1-U}$$

Or

$$Q_{n+1} > \frac{K+Q_1+UQ_2+U^2Q_3+...+U^{n-1}Q_n}{1+U+U^2+...+U^{n-1}} \qquad (20)$$

Or

$$Q_{n+1} > \frac{K+\sum_{r=1}^{n}Q_rU^{r-1}}{\sum_{r=1}^{n}U^{r-1}} \qquad (21)$$

This implies that the next period cost is greater than the weighted average of previous costs, because the expression on the right-hand side of the equation (19) is the weighted average of all costs up to and including period n – l. The weighted averages $1, U, U^2...,U^{n-1}$ are the discount factors associated with each period's cost.

Similarly, we can express the left-hand side of the equation (19) as follows:

$$Q_n < \frac{K+Q_1+UQ_2+U^2Q_3+...+U^{n-1}Q_n}{1+U+U^2+...+U^{n-1}} \qquad (22)$$

Or

$$Q_n < \frac{K+\sum_{r=1}^{n}Q_rU^{r-1}}{\sum_{r=1}^{n}U^{r-1}} \qquad (23)$$

Using the mathematical expression in Equations (20) and (22), we make the following optimal policies:

a) Replace the vehicle if the cost of the next period is greater than the weighted average of the previous costs.

b) Do not replace the vehicle if the cost of the next period is less than the weighted average of the previous costs.

c) We can obtain from Equation (14) the corresponding value of the minimum yearly payment y, that is

$$y = \frac{1-U}{1-U^n}P_n$$

. Moreover, if y_1 and y_2 are the minimum yearly payments associated with two vehicles, say A and B, vehicle A will be chosen if $Y_1 < y_2$ and vice versa. It should be noted that as the interest rate j tends to zero $(j \to o),$ discount rate u tends to one, and the expression stated in Equation (20) reduces to

$$Q_{n+1} > \frac{K+Q_1+Q_2+..Q_n}{1+1+1+....+n\ times} \qquad (24)$$

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RESULTS AND DISCUSSION
An automobile driver observes from his records that the cost per year of maintaining his car, which he purchased at the cost of N7 million, is as follows:

Year	1	2	3	4	5	6	7	8
Running cost (N)	1.1	1.3	1.5	1.9	2.4	2.9	3.5	4.1
Resale price(N)	3.1	1.6	.85	.475	.30	.30	.30	.30

(a) Determine the age at which the vehicle replacement will be due.
(b) Another automobile driver has three cars with the same purchase price and cost of maintaining each as in part (a). Two of these cars are 2 years old, and the third one is a year old. He is thinking of buying a new type of car, which is 50% higher in performance than one of the old cars, and at a unit price of N9 million. The running costs and resale price for the new car is estimated in millions of naira as follows:

Year	1	2	3	4	5	6	7	8
Running cost (N)	1.6	1.3	1.9	2.5	3.2	4.1	5.1	6.2
Resale Price(N)	4.100	2.100	1.100	.600	.400	.400	.400	.400

Assuming that the loss of flexibility due to fewer cars is of no effect, and that he will continue to have sufficient work for three of the old cars, what should be his optimal replacement policy? **Results**
The average annual cost for the old car is computed in the table below using the procedure outlined in our proposed models.

Table 1: Average Annual Cost of the Old Car

(1) Years of service (n)	(2) Resale value (H) (N)	(3) Purchase price-resale value (K-H) (N)	(4) Annual maintenance cost Z(t) (N)	(5) Summation of maintenance cost $\sum_{0}^{n} Z(t)$ (N)	(6) Total Cost (3) + (5) (N)	(7) Average annual cost $\frac{(6)}{(1)}$ (N)
1	3.100	3.900	1.100	1.100	5.000	5.000
2	1.600	5.400	1.300	2.400	7.800	3.900
3	.850	6.150	1.500	3.900	10.050	3.350
4	.475	6.525	1.900	5.800	12.325	3.081
5	.300	6.700	2.400	8.200	14.900	2.980
6	.300	6.700	2.900	11.100	17.800	2.967
7	.300	6.700	3.500	14.600	21.300	3.043
8	.300	6.700	4.100	18.700	25.400	3.175

Thus, the replacement of the old car should be done at the end of the 6th year.

(a) To compute the average yearly cost of the new car whose performance is higher, we use the table below as follows:

Table 2: Average Annual Cost of the New Car

(1) Years of service (n)	(2) Resale value (H) (N)	(3) Purchase price-resale value (K-H) (N)	(4) Annual maintenance cost Z(t) (N)	(5) Summation of maintenance cost $\sum_{0}^{n} Z(t)$ (N)	(6) Total Cost (3) + (5) (N)	(7) Average annual cost $\frac{(6)}{(1)}$ (N)		
1	4.100	4.900	1.300	1.300	6.200	6.200		
2	2.100	6.900	1.600	2.900	9.800	4.900		
3	1.100	7.900	1.900	4.800	12.700	4.233		
4	.600	8.400	2.500	7.300	15.700	3.925		
5	.400	8.600	3.200	10.500	19.100	3.820		
6	.400	8.600	4.100	14.600	23.200	3.867		

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7	.400	8.600	5.100	19.700	28.300	4.043	
8	.400	8.600	6.200	25.900	34.500	4.312	

old cars with the two new ones in the third year.

From Table 1, the replacement of the old car should be done at the end of the 6th year. This is because the average annual cost of maintenance in the 7th year (N3.043m) is higher than the average annual cost of maintenance in the 6th year (N2.967m). To determine his optimal replacement policy, assuming that the loss of flexibility due to fewer cars is of no effect, and that he will continue to have sufficient work for three of the old cars, we use the calculations in Table 2. In Table 2, we observed that the average annual cost of maintenance in the 6th year (N3.867m) is higher than the average annual cost of maintenance in the 5th year (N3.820m). Hence, replacement should be done in the 5th year. Since the new car is 50% higher in terms of performance than the old one, the minimum average annual maintenance cost of N3.820m for the former car is equivalent to $N3.820 \times \frac{2}{3} = $ N2.547m for the latter. Since his amount is less than N2.967m for it, the latter will be replaced by the new car.

Since the decision is to replace the old car with the new one, the next step is to determine when this replacement should be done. For the sake of uniformity, we assume that all three old cars will be replaced by two new ones with higher performance. The new cars will be purchased when the cost for the following year of using the three old cars becomes more than the average yearly cost of the two new ones.

Total annual cost of one car with lower performance during the first year = N5.000m
Annual cost of one car with lower performance during the second year
$$= N7.800m - N5.000m = N2.800m,$$
Annual cost of one car with lower performance during the third year = N2.250m,
Annual cost of one car with lower performance during the fourth year = N2.275m,
Annual cost of one car with lower performance during the fifth year = N2.575m,
Annual cost of one car with lower performance during the sixth year $N900m,$ and so on.
Total cost during the first year for two cars with lower performance aged two years, and one vehicle aged one year $= 2 \times N2.250m + N2.800m = N7.300m.$
Total cost during the following second year $= 2 \times N2.275m + N2.250m = N6.800m,$
Total cost during the following third year $= 2 \times N2.575m + N2.275m = N7.425m.$
Total cost during the following fourth year $= 2 \times N2.900m + 2.575m = N8.375,$ and so on.
But the minimum average cost for two new cars $= 2 \times N3.820m = N7.640m.$
Since the total cost of old cars during the next third year is lower than the minimum average cost of the new cars and becomes higher only in the next fourth year, the old cars should be replaced by the new cars whose performance is higher after the next third year of their usage. Thus, the optimal policy is to replace the three

Conclusion
In this paper, we have expanded earlier work on replacement models which uses dynamic programming technique to analyze the impact of depreciation of state-owned vehicles by applying the finite difference equations approach. The aim is to find the optimum replacement time of automobile vehicles whose economic values deteriorate with time, thereby causing an increase in maintenance and operating costs. In model 1, we considered a situation that involves the replacement of vehicles whose costs of repair and maintenance vary with time. The objective of the model is to minimize the average yearly cost of operating vehicles whose costs of maintenance increase with time, while the scrap value is constant. It is assumed that the time value of money is negligible with a zero-interest rate. Hence, the calculation is based on the annual average cost. In model 2, we discussed the replacement of vehicles that deteriorate with time, thereby causing an increase in maintenance and operational costs, and the time value of money is assumed to vary.

It is observed from the numerical illustration of the models that the optimal life of a vehicle can be determined by comparing the increased running costs to decreased depreciation values. The two models presented have a computational advantage over existing ones in terms of assisting policy makers to determine vehicles' optimal replacement policy. The policies include deciding which vehicle should be repaired or replaced, thereby averting losses of human and material resources due to accidents.

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