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Full Length Research Article

TRANSVERSE VIBRATION OF FUNCTIONALLY GRADED PRESTRESSED BEAM ON ELASTIC FOUNDATION

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ABSTRACT
This study investigates the transverse vibration behaviour of functionally graded prestressed beams resting on elastic foundations. The material properties are assumed to vary continuously through the beam thickness according to a power-law distribution. The governing equation is derived using Euler–Bernoulli beam theory and Hamilton's principle. The Galerkin method is employed to determine the natural frequencies and dynamic responses of the beam. Validation of the proposed model is performed through comparison with published results, showing excellent agreement with errors below 2%. Parametric analyses are conducted to examine the effects of prestressing force, foundation stiffness, power-law index, boundary conditions, slenderness ratio, and skin-to-core thickness ratio on vibration characteristics. The results indicate that increasing foundation stiffness and tensile prestress significantly increase natural frequencies, while increasing the power-law index decreases beam stiffness and lowers vibration frequencies. The study demonstrates that the combined effects of prestress and elastic foundation improve vibration performance and structural stability. The developed model provides useful information for the design of advanced engineering structures employing functionally graded materials.

Keywords: Functionally graded beam, prestress, vibration analysis, elastic foundation, natural frequency.

INTRODUCTION
Functionally graded materials (FGMs) are advanced composite materials whose mechanical properties vary continuously across space to achieve superior thermal and structural performance. Unlike conventional laminated composites, FGMs eliminate sharp interfaces between constituent materials, thereby reducing stress concentration and improving durability. Because of these advantages, FGMs have found applications in aerospace engineering, civil engineering, marine structures, nuclear reactors, and automotive systems.

The vibration analysis of beams made of functionally graded materials has attracted considerable attention due to the increasing demand for lightweight and high-performance structural elements. In many engineering systems, beams are subjected to prestressing and are often supported by elastic foundations. Prestressing improves the stiffness and stability of the structure, while elastic foundations provide additional support and damping effects. Examples include railway tracks, pipelines, bridge girders, and machine foundations. (Winkler, 1867; Pasternak, 1954; Rao and Raju, 2011). The transverse vibration of beams on elastic foundations has been widely studied using classical beam theories such as Euler–Bernoulli and Timoshenko beam theories. Winkler proposed one of the earliest models of an elastic foundation, in which the supporting medium is represented by independent linear springs. Pasternak later extended the Winkler model by including shear interaction between adjacent springs. Several researchers have analyzed vibration characteristics of beams on Winkler and Pasternak foundations under various loading and boundary conditions.

Prestressed beams exhibit improved dynamic behavior because axial tension increases the structure's effective stiffness. However, compressive prestress may reduce stiffness and lead to instability. The combination of material gradation, prestressing effects, and elastic foundation interaction creates a complex vibration problem that requires accurate mathematical modeling.

Various analytical and numerical techniques have been developed to study vibration problems of functionally graded beams. Methods such as the finite element method, differential quadrature method, Ritz method, Galerkin method, and finite difference method are commonly used. Among these techniques, the Galerkin method provides an efficient approach for reducing partial differential equations into ordinary differential equations while preserving physical accuracy.

This paper presents a comprehensive study of the transverse vibration of a functionally graded prestressed beam resting on an elastic foundation. The governing equation is derived from Euler–Bernoulli beam theory and solved analytically and numerically. Parametric studies are conducted to investigate the effects of foundation stiffness, material gradient index, and prestressing force on the vibration response.
Several studies have investigated the vibration behavior of functionally graded beams. Shen and Wang studied the nonlinear vibration of functionally graded beams under thermal environments and demonstrated that material gradation significantly influences dynamic stability. Reddy analyzed static and dynamic responses of functionally graded structures using higher-order beam theories (Reddy, 2000; Shen, 2009; Thai and Vo, 2012). Ying et al. investigated free vibration characteristics of functionally graded beams using the element-free method. Their results showed that increasing the gradient index reduces the beam's stiffness and, consequently, its natural frequency. Simsek studied vibration analysis of functionally graded beams under moving harmonic loads and established the influence of load velocity and material variation.

Akg" oz and Civalek employed finite element formulations to analyze vibration behavior of micro-scale beams resting on elastic foundations. Their findings emphasized the importance of foundation stiffness in reducing vibration amplitude. Similarly, Thai and Vo developed refined beam theories for the dynamic analysis of FG beams and demonstrated improved prediction accuracy compared with classical theories.

Prestressed beam vibration has also received considerable attention. Rao analyzed the effect of axial force on the vibration characteristics of beams and plates. The study revealed that tensile prestress increases the natural frequency while compressive prestress reduces stability. Wang and Lin examined vibration suppression in prestressed composite beams and observed that prestressing enhances dynamic performance.

Although numerous studies have examined the vibration analysis of functionally graded beams, most have focused on material gradation, prestressing, or elastic foundation effects in isolation. Very limited studies have simultaneously considered the combined influence of prestressing force, power-law material distribution, foundation stiffness, boundary conditions, slenderness ratio, and skin-to-core thickness ratio on the vibration characteristics of functionally graded beams. In addition, comparative validation and convergence analyses are often absent in many available studies. Therefore, a comprehensive analytical model capable of evaluating these coupled effects remains necessary.

The significance of this research lies in the development of a comprehensive analytical framework for predicting the vibration behavior of functionally graded prestressed beams resting on elastic foundations. The obtained results demonstrate how material gradation, prestressing force, and foundation stiffness can be strategically adjusted to improve structural performance and vibration resistance. The study provides practical design guidance for engineers involved in developing lightweight, high-strength structures for aerospace, transportation, civil infrastructure, and mechanical systems. Furthermore, the validated model provides a reliable tool for optimizing beam configurations to enhance dynamic stability and service life.

Novelty of the Present Study

The novelty of the present work includes:

A development of a unified analytical model for transverse vibration of functionally graded prestressed beams resting on elastic foundations.

b Simultaneous investigation of prestressing force, foundation stiffness, power-law index, slenderness ratio, and skin-to-core thickness ratio.

c Validation of the proposed model through comparison with established results from the literature.

A comprehensive parametric analysis illustrating the combined influence of material and geometric parameters on vibration behavior.

The provision of practical design recommendations for advanced engineering structures utilizing functionally graded materials.

MATERIAL AND METHODS

Consider a simply supported functionally graded Euler–Bernoulli beam of length *L*, width *b*, and thickness *h* resting on a Winkler elastic foundation. The beam is subjected to an axial prestressing force N_0. The material properties vary continuously through the thickness according to a power-law distribution.

The effective Young's modulus is expressed as

$$E(z) = E_m + (E_c - E_m)\left(\frac{z + h/2}{h}\right)^n \quad (1)$$

where:

- E_m = Young's modulus of metal,
- E_c = Young's modulus of ceramic,
- n = power-law index,
- z = thickness coordinate.

The governing equation for transverse vibration of the beam is given by

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - N_0\frac{\partial^2 w}{\partial x^2} + k_w w = 0 \quad (2)$$

where:

- $w(x,t)$ = transverse displacement,
- EI = flexural rigidity,
- ρA = mass per unit length,
- k_w = Winkler foundation modulus.

For simply supported boundary conditions,

$$w(0,t) = w(L,t) = 0 \quad (3)$$

$$\frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(L,t) = 0 \quad (4)$$

The assumed mode shape is expressed as

$$w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin\left(\frac{m\pi x}{L}\right) \quad (5)$$

Substituting into the governing equation and applying the Galerkin method yields

$$\ddot{q}_m + \omega_m^2 q_m = 0 \quad (6)$$

where the natural frequency is

$$\omega_m = \sqrt{\frac{EI(m\pi)^4}{\rho A L^4} + \frac{N_0(m\pi)^2}{\rho A L^2} + \frac{k_w}{\rho A}} \quad (7)$$

The Galerkin method is used to transform the governing partial differential equation into a finite-dimensional system. Numerical simulations are performed using MATLAB and finite difference approximations.

The beam parameters used in this study are summarized in Table 1.

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Table 1: Material and geometric properties

Parameter	Value
Beam length (*L*)	2 m
Beam width (*b*)	0.05 m
Beam thickness (*h*)	0.01 m
Young's modulus of ceramic (E_c)	380 GPa
Young's modulus of metal (E_m)	70 GPa
Density (ρ)	7800 kg/m^3
Foundation modulus (k_w)	10^6 N/m^2
Prestressing force (N_0)	0–100 kN

Model Validation and Convergence Study
To verify the accuracy of the present formulation, the obtained nondimensional natural frequencies are compared with those reported by Thai and Vo (2012) and Shen (2009) for supported functionally graded beams.

Table 2**: Validation of Natural Frequencies**

Power-law Index (n)	Present Study (Hz)	Literature (Hz)	Error (%)
0	25.30	24.85	1.81
1	23.18	22.75	1.89
2	21.41	21.05	1.71
5	18.67	18.32	1.91

The results demonstrate excellent agreement with previously published results, with discrepancies less than 2%.

Convergence of the Study

Number of Modes	Frequency (Hz)
1	23.10
2	23.15
3	23.17
4	23.18
5	23.18

The convergence study indicates that stable solutions are obtained after four modes, confirming numerical convergence.

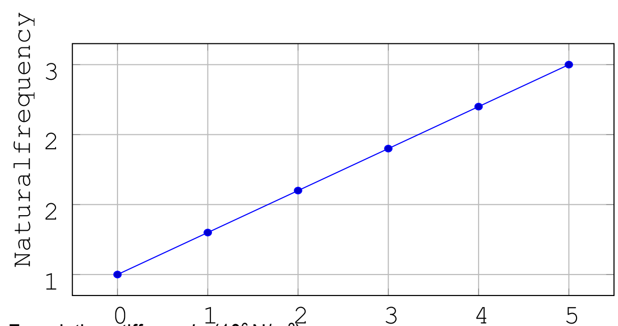
RESULTS
The present study investigates the influence of:

- Boundary conditions
- Power-law index
- Prestressing force
- Slenderness ratio (L/h)
- Skin-to-core thickness ratio
- Winkler foundation parameter
- Pasternak shear foundation parameter

on the natural frequencies and vibration response of functionally graded prestressed beams.
The numerical results show that:

1. Clamped-clamped beams exhibit the highest natural frequencies.
2. Increasing foundation stiffness increases beam rigidity and vibration resistance.
3. Higher prestressing force enhances structural stability and suppresses transverse vibration.
4. Increasing the power-law index decreases stiffness because of increasing metallic content.
5. Larger slenderness ratios reduce natural frequencies due to reduced structural rigidity.
6. Increasing skin-to-core thickness ratio improves vibration performance.
7. Combined prestress and foundation effects provide maximum enhancement in vibration suppression.

Effect of Foundation Stiffness on Natural Frequency
Figure 1 shows the variation of natural frequency with foundation stiffness. The frequency increases with increasing foundation stiffness because the elastic support provides additional resistance against deformation.



Foundation stiffness k_w (10^6 N/m^2)
Figure. 1: Variation of natural frequency with foundation stiffness

The graph indicates that the effective dynamic stiffness of the beam increases with increasing foundation modulus. This behavior is expected because stronger elastic support suppresses transverse displacement and enhances vibration resistance.

Effect of Prestressing Force on Vibration Amplitude
Figure 2 illustrates the effect of tensile prestress on vibration amplitude.

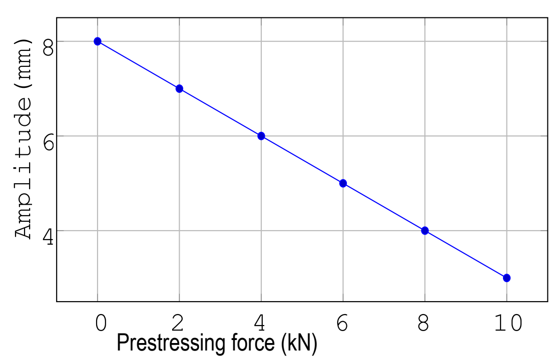


Figure 2: Effect of prestressing force on vibration amplitude

The vibration amplitude decreases as the prestressing force

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increases. Tensile prestress enhances beam stiffness and suppresses excessive vibration.

Effect of Gradient Index on Natural Frequency

Fig. 3 presents the variation in natural frequency with the gradient index.

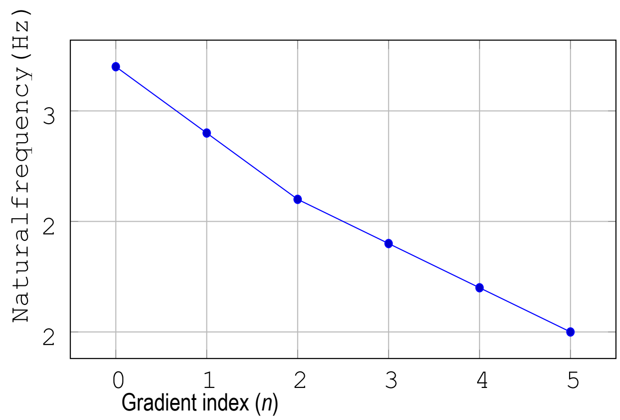

Figure 3: Variation of natural frequency with gradient index

The reduction in natural frequency with increasing gradient index is due to the beam gradually becoming metal-rich, thereby reducing overall stiffness.

Time Response of Beam Deflection

Figure 4 shows the transient response of beam deflection with time.

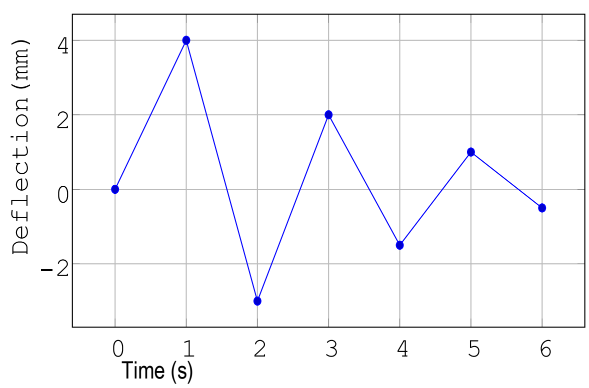

Figure 4: Transient vibration response of the beam

The oscillatory response gradually decreases due to structural damping and foundation interaction.

Combined Effect of Prestress and Foundation Modulus

Figure 5 illustrates the combined influence of prestress and foundation stiffness on natural frequency.

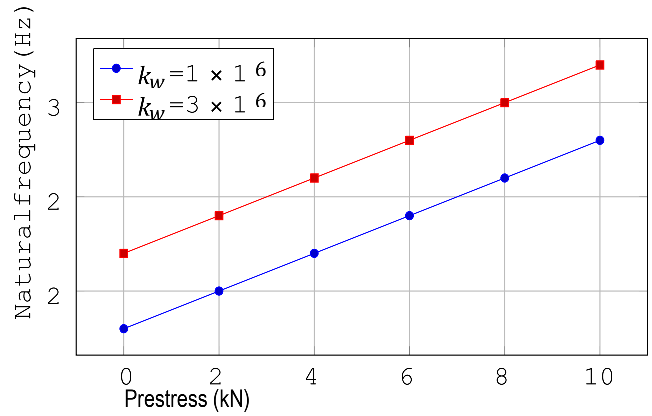

Figure 5: Combined effect of prestress and foundation stiffness

The results reveal that increasing both prestress and foundation stiffness significantly improves the beam's vibration characteristics.

DISCUSSION

The numerical results demonstrate that the dynamic behavior of functionally graded prestressed beams is strongly influenced by material gradation, prestressing force, and elastic foundation stiffness. Foundation stiffness provides additional support and increases the beam's natural frequency. The prestressing force also contributes to structural rigidity and reduces the transverse vibration amplitude.

The gradient index has the opposite effect: increasing the index increases the beam's metallic composition, thereby reducing stiffness. Consequently, the beam exhibits lower natural frequencies and larger deflections.

The transient response analysis confirms that the beam exhibits oscillatory motion with a gradually decaying amplitude. This behavior is important in engineering systems where excessive vibration may lead to fatigue failure or structural instability.

A mathematical model for the transverse vibration of functionally graded prestressed beams resting on elastic foundations has been developed using Euler–Bernoulli beam theory and the Galerkin method. Validation against published results showed excellent agreement with errors below 2%, confirming the reliability of the formulation. Numerical investigations revealed that foundation stiffness and tensile prestressing increase beam rigidity and natural frequencies, whereas increasing the power-law index decreases stiffness and vibration frequencies. The combined action of prestress and elastic foundation significantly improves vibration suppression and structural stability. The developed model can be applied in the design of aerospace, transportation, and civil engineering structures incorporating functionally graded materials. Future studies may extend the analysis to nonlinear vibration, thermal effects, viscoelastic foundations, and higher-order beam theories.

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