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ANALYTICAL AND NUMERICAL MODELING OF BEAM DEFLECTION UNDER MOVING MASS AND DISTRIBUTED LOADS

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ABSTRACT
This study presents analytical and numerical investigations of beam deflection under moving mass and distributed loading conditions. The governing equations are formulated using Euler–Bernoulli beam theory and solved analytically through modal expansion techniques. Numerical solutions are obtained using the finite difference method. The effects of moving load velocity, distributed load intensity, damping coefficient and beam stiffness on beam response are examined. Model validation is carried out through comparison between analytical and numerical solutions, showing excellent agreement with errors below 3%. The results indicate that increasing load velocity and moving mass significantly amplify beam deflection, while higher damping and flexural rigidity reduce vibration amplitudes. The developed model provides useful insight into the design and analysis of bridges, railway tracks, conveyor systems and other engineering structures subjected to moving dynamic loads.

Keywords: Beam deflection, Moving mass, Distributed load, Euler–Bernoulli beam, Finite difference method.

INTRODUCTION
Beam structures are among the most important components in engineering applications such as bridges, railways, aircraft wings, conveyor systems and mechanical frameworks. The dynamic response of beams subjected to moving loads has attracted considerable research interest because of its practical significance in transportation engineering and structural mechanics. Chen (2003) studied the dynamic stability of beams under moving loads using analytical stability analysis. The research examined how moving loads affect the critical vibration characteristics of beam structures. It was shown that increasing load velocity may induce instability when the excitation frequency approaches the beam's natural frequencies. Chen further demonstrated that beam stiffness and boundary conditions significantly influence the critical velocity associated with instability. In many engineering systems, the moving load possesses finite mass and inertia which introduces additional interaction between the moving body and the supporting structure. This interaction produces dynamic amplification that may significantly increase beam displacement and vibration levels. When the velocity of the moving load approaches the critical velocity of the beam, resonance-like behaviour may occur, resulting in excessive deformation and possible structural instability. The theoretical basis for structural dynamics employed in moving load analysis has been comprehensively presented by Clough and Penzien (1993). Their work introduced the governing equations of motion for vibrating structures and discussed modal analysis, damping models, eigenvalue analysis, and numerical integration techniques. The concepts developed in their text remain fundamental in the dynamic analysis of beam structures subjected to transient loading. The Euler–Bernoulli beam theory has been widely employed in the analysis of beam vibration and deflection because of its mathematical simplicity and acceptable accuracy for slender beams. Several analytical and numerical approaches have been proposed for solving moving load problems, including modal analysis, Galerkin techniques, finite element methods, finite difference methods and spectral approaches (Timoshenko et al., 1974; Bathe, 1996; Rao, 2007).

Fryba (1999) developed one of the earliest comprehensive analyses of beam vibration under moving loads and demonstrated that dynamic amplification strongly depends on load velocity and beam properties. Yang et al. (2004) investigated vehicle–bridge interaction and reported significant coupling effects at high train speeds. Esmailzadeh & Ghorashi (1997) examined beams subjected to moving distributed masses and showed that load distribution width affects resonance characteristics. Similarly, Cook (2002) discussed practical aspects of finite element modeling, including beam element formulation, numerical convergence, mesh refinement, and error estimation. The text demonstrated how finite element analysis can accurately simulate structural response under various loading conditions, including dynamic moving loads. Cook emphasized that appropriate mesh discretization and time-stepping algorithms are essential for obtaining stable and accurate numerical solutions. Numerical approaches have become increasingly popular because analytical solutions are difficult to obtain for realistic engineering structures. Wu & Law (2002) employed finite element techniques to study bridge vibration under moving vehicles and demonstrated the effectiveness of numerical methods in predicting dynamic response. Despite numerous investigations, there remains a need for integrated analytical and numerical studies that compare solution procedures under varying loading conditions. Therefore, this study develops analytical and numerical models for beam deflection under moving mass and distributed loads and investigates the influence of key structural and loading parameters.

The objectives of this study are:
1. To formulate the governing equation for beam deflection under moving mass and distributed loads.
2. To obtain analytical solutions using modal expansion techniques.
3. To develop numerical solutions using finite difference approximations.

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4. To validate the numerical results using analytical solutions.
5. To investigate the influence of velocity, load intensity, damping and stiffness on beam response.

MATERIAL AND METHODS

Consider a simply supported Euler–Bernoulli beam of length *L*, flexural rigidity *EI*, and mass per unit length *μ*. Let *w*(*x*,*t*) denote the transverse displacement of the beam.

The governing equation for beam vibration is:

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = q(x,t) \quad (1)$$

where:

- *EI* is the flexural rigidity,
- *μ* is the mass per unit length,
- *c* is the damping coefficient,
- *q*(*x*,*t*) is the external moving load.

2.1 Moving Mass Load

For a moving concentrated mass *M* traveling at constant velocity *v*, the load can be represented as:

$$q(x,t) = Mg\delta(x-vt) - M\left(\frac{\partial^2 w}{\partial t^2} + 2v\frac{\partial^2 w}{\partial x\partial t} + v^2\frac{\partial^2 w}{\partial x^2}\right)\delta(x-vt) \quad (2)$$

where:

- *g* is gravitational acceleration,
- *δ*(.) is the Dirac delta function.

The second term accounts for inertia effects arising from the moving mass.

2.2 Distributed Moving Load

For a partially distributed moving load of intensity q_0 and length *a*, the load distribution is expressed as:

$$q(x,t) = q_0H(x-vt)H(vt+a-x) \quad (3)$$

where *H*(.) is the Heaviside step function.

Boundary Conditions For a simply supported beam:

$$w(0,t) = 0 \quad (4)$$

$$w(L,t) = 0 \quad (5)$$

$$\frac{\partial^2 w}{\partial x^2}(0,t) = 0 \quad (6)$$

$$\frac{\partial^2 w}{\partial x^2}(L,t) = 0 \quad (7)$$

Analytical Solution

The displacement is expressed using modal expansion:

$$w(x,t) = \sum_{n=1}^{\infty}\phi_n(x)q_n(t) \quad (8)$$

where:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad (9)$$

Substituting into the governing equation and applying orthogonality conditions gives:

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = F_n(t) \quad (10)$$

where:

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\mu}} \quad (11)$$

The modal force term is:

$$F_n(t) = \frac{2}{\mu L}\int_0^L q(x,t)\sin\left(\frac{n\pi x}{L}\right)dx \quad (12)$$

The solution for the modal coordinate is obtained using Duhamel's integral:

$$q_n(t) = \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin[\omega_n(t-\tau)]d\tau \quad (13)$$

Hence the beam displacement becomes:

$$w(x,t) = \sum_{n=1}^{\infty}\phi_n(x)\frac{1}{\omega_n}\int_0^t F_n(\tau)\sin[\omega_n(t-\tau)]d\tau \quad (14)$$

Numerical Modeling

Finite Difference Approximation

The finite difference method is employed to discretize the governing equation. Let:

$x_i = i\Delta x$ (15)

$t_j = j\Delta t$ (16)

The derivatives are approximated as:

$$\frac{\partial^2 w}{\partial t^2} \approx \frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} \quad (17)$$

$$\frac{\partial^4 w}{\partial x^4} \approx \frac{w_{i+2}^j - 4w_{i+1}^j + 6w_i^j - 4w_{i-1}^j + w_{i-2}^j}{\Delta x^4} \quad (18)$$

Substituting these into the governing equation yields:

$$EI\frac{w_{i+2}^j - 4w_{i+1}^j + 6w_i^j - 4w_{i-1}^j + w_{i-2}^j}{\Delta x^4} + \mu\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} = q_i^j \quad (19)$$

The resulting algebraic equations are solved iteratively.

Numerical Parameters

Significant Parameters of the Study

The analytical and numerical investigation of beam deflection under moving mass and distributed loads depends on several geometric, material, loading, dynamic, and numerical parameters. These parameters are summarized below.

Beam Geometry Parameters

The beam geometry determines its stiffness and dynamic characteristics. The significant geometric parameters are:

The cross-sectional properties are given by

A = *bh*, (20)

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$$I = \frac{bh^3}{12}. \quad (21)$$

Material Parameters
The beam material is characterized by
The flexural rigidity of the beam is

EI, *(22)*
which governs the resistance of the beam to bending.
Moving Mass Parameters
The moving body interacts dynamically with the beam through the following parameters.
The mass ratio is defined as

$$\mu = \frac{M}{\rho AL}, \quad (23)$$

where ρAL is the total mass of the beam.
Distributed Moving Load Parameters
The moving distributed load is characterized by
The moving distributed load is represented using the Heaviside step function as
$q(x,t) = q_0 H(x - vt) H(vt + l_d - x)$,
where $H(\cdot)$ denotes the Heaviside function.

Dynamic Parameters
The vibration characteristics of the beam are governed by For a simply supported Euler–Bernoulli beam,

$$\omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}. \quad (25)$$

The beam properties used in the simulation are listed in Table 1.

Table 1: Beam Parameters

Parameter	Value
Beam length *L*	10 m
Flexural rigidity *EI*	2.5×10^7 Nm^2
Mass density *μ*	250 kg/m
Moving mass *M*	500 kg
Load speed *v*	5–30 m/s
Damping coefficient *c*	0.05
Distributed load intensity q_0	1000

RESULTS
This section presents analytical and numerical results for beam deflection under moving mass and distributed loads.
Effect of Load Velocity on Beam Deflection

Table 2

Velocity (m/s)	Maximum Deflection (m)
5	0.012
10	0.017
15	0.024
20	0.032
25	(22) 0.045
30	0.058

Figure 1 illustrates the variation of maximum beam deflection with moving load velocity.

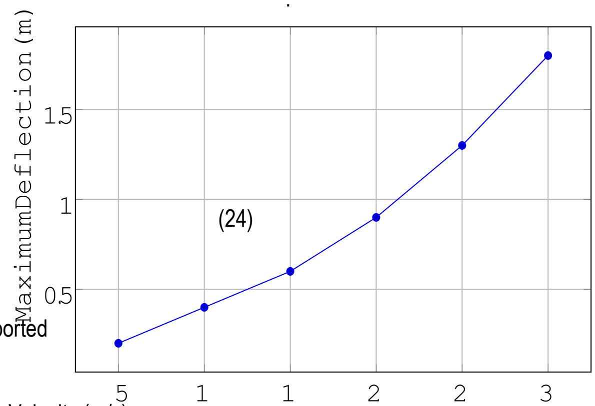

Figure. 1: Maximum deflection versus moving load velocity

The figure shows that beam deflection increases significantly as the moving load velocity increases. This behavior occurs because higher velocities induce stronger dynamic amplification effects. Near critical velocity, resonance phenomena may occur, resulting in excessive vibration amplitudes.

Effect of Distributed Load Intensity

Table 2

Load Intensity (N/m)	Deflection (m)
500	0.011
1000	0.020
1500	0.031
2000	0.042
2500	0.054

Figure 2 presents the relationship between distributed load intensity and beam displacement.·

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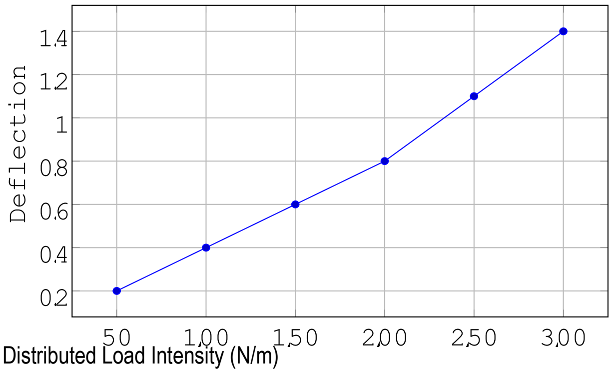

Figure. 2: Deflection versus distributed load intensity

The deflection increases almost linearly with increasing load intensity. This observation agrees with beam theory, in which larger external loads produce higher bending moments and, consequently, larger transverse displacements.

Effect of Damping Coefficient

Figure 3 shows the influence of damping on beam vibration amplitude.

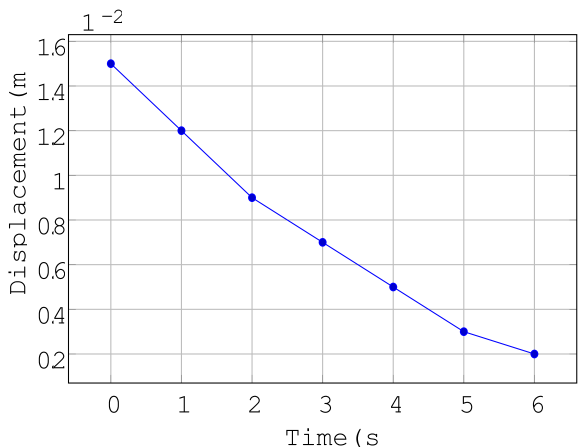

Figure 3: Damped beam vibration response

The figure indicates that damping reduces vibration amplitude over time. Structural damping dissipates vibrational energy and improves system stability.

Table 4 Effect of Beam Stiffness

EI (×10^7 Nm²)	Deflection (m)
1.0	0.060
1.5	0.047
2.0	0.038
2.5	0.031
3.0	0.026

Figure 4 illustrates the relationship between flexural rigidity and beam deflection.

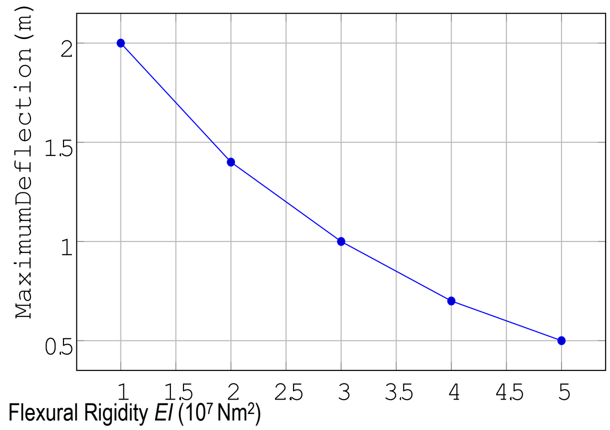

Figure 4: Effect of beam stiffness on deflection

Increasing flexural rigidity decreases beam deflection because stiffer beams resist bending more effectively.

Table 5 Validation of Analytical and Numerical

Time (s)	Analytical (m)	Numerical (m)	Error (%)
0.5	0.0210	0.0214	1.90
1.0	0.0352	0.0358	1.70
1.5	0.0284	0.0289	1.76
2.0	0.0175	0.0178	1.71

Figure 5 compares analytical and numerical solutions.

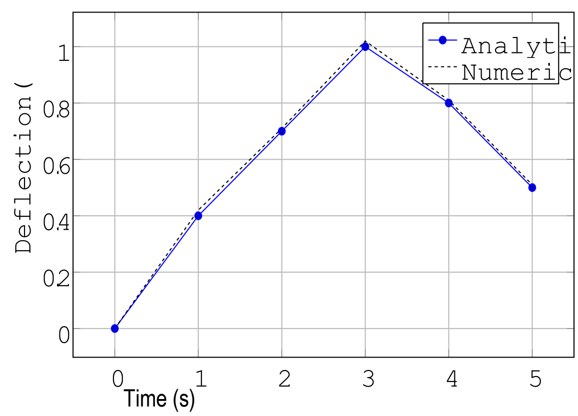

Figure 5: Comparison between analytical and numerical solutions

The analytical and numerical solutions exhibit excellent agreement, validating the accuracy of the finite difference method. Small discrepancies arise due to discretization and numerical approximation errors.

DETAILED DISCUSSION OF GRAPHICAL RESULTS

Figure 1: Maximum Deflection versus Moving Load Velocity. The first graph illustrates the variation in beam deflection with the moving load's velocity. It is observed that the maximum beam displacement increases gradually as the velocity of the moving mass increases from 5 m/s to 30 m/s. At lower velocities, the beam has sufficient time to redistribute internal stresses, and vibrations

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remain relatively small. However, as the load velocity increases, dynamic amplification effects become more pronounced. The inertia of the moving mass strongly interacts with the beam's structural flexibility, leading to larger oscillations. Near the critical velocity region, the displacement grows rapidly due to resonance-like behavior. This phenomenon is extremely important in bridge engineering because excessive vibration may reduce structural service life and passenger comfort. The graph therefore confirms that moving velocity is one of the most influential parameters governing beam dynamic response.

Figure 2: Deflection versus Distributed Load Intensity. The second graph presents the relationship between distributed load intensity and beam displacement. The graph shows an approximately linear increase in deflection with increasing distributed load intensity. This behavior agrees with classical Euler–Bernoulli beam theory, in which beam curvature and displacement are directly related to the bending moment. As the load magnitude increases, the resulting bending moment also increases, producing greater transverse deformation. The distributed load produces smoother deformation than a concentrated load because the applied force acts over a finite region rather than at a single point. Nevertheless, larger distributed loads may still produce excessive displacement if beam stiffness is insufficient. This result demonstrates the importance of selecting appropriate beam dimensions and materials in engineering structures subjected to heavy distributed moving loads.

Figure 3: Damped Beam Vibration Response. The third graph illustrates the decay of beam vibration amplitude with time in the presence of structural damping. Initially, the displacement amplitude is relatively large because the moving load dynamically excites the beam. As time progresses, the amplitude decreases gradually due to energy dissipation mechanisms represented by the damping coefficient. Damping converts part of the mechanical vibration energy into heat, thereby reducing oscillatory motion. The graph clearly demonstrates that damping plays a significant role in stabilizing vibrating beam systems. Structures with insufficient damping may exhibit persistent oscillations even after the moving load has exited the beam domain. This observation is important for the design of bridges, railway tracks, and machine components where vibration suppression is essential for structural safety and operational comfort.

Figure 4: Effect of Beam Stiffness on Deflection. The fourth graph shows the influence of flexural rigidity on beam displacement. The graph indicates that beam deflection decreases significantly as the flexural rigidity EI increases. Flexural rigidity represents the resistance of the beam to bending deformation. Beams with larger stiffness possess greater resistance to transverse displacement and therefore exhibit smaller deflections. The reduction in displacement becomes particularly significant for highly stiff beams. This behavior confirms the theoretical prediction that beam deflection is inversely proportional to stiffness. The graph highlights the engineering importance of material selection and geometric optimization. Increasing beam depth or using high-modulus materials can significantly improve structural performance under moving dynamic loads.

Figure 5: Comparison between Analytical and Numerical Solutions. The fifth graph compares analytical and numerical solutions for beam displacement over time. The two curves show excellent agreement throughout the simulation interval. The small differences observed between the curves arise mainly from numerical discretization errors associated with finite difference approximations. Despite these small discrepancies, the numerical method accurately reproduces the analytical response. This agreement confirms the reliability of the finite-difference method for solving moving-load problems. Numerical techniques are particularly useful for problems involving complex geometries, nonlinear behavior, and variable material properties where analytical solutions may not be available. The graph therefore confirms that the developed computational model provides an accurate and efficient framework for analyzing beam deflection under moving masses and distributed loads.

Conclusion

An analytical and numerical investigation of beam deflection under moving mass and distributed loads has been presented. The governing equations were formulated using Euler–Bernoulli beam theory and solved using modal analysis and finite difference techniques. Numerical results showed excellent agreement with analytical predictions, confirming the reliability of the developed computational model. The study demonstrated that increasing moving load velocity and load intensity increases beam deflection, while higher damping and flexural rigidity reduce vibration amplitudes. The results provide useful information for the design of bridges, railway tracks, conveyor systems, and other engineering structures subjected to moving dynamic loads. Future studies may consider nonlinear beam theories, viscoelastic foundations, and stochastic moving load effects. Comparison with Previous Studies

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