

DEVELOPMENT OF THE WEIBULL EXPONENTIATED FRÉCHET DISTRIBUTION (WEFD): STATISTIC AND APPLICATIONS

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ABSTRACT

This research introduced and studied the Weibull Exponentiated Fréchet Distribution (WEFD), a new five-parameter continuous probability distribution constructed by applying the Weibull exponentiated generator to the classical Fréchet baseline. The proposed model is designed to accommodate a broad range of distributional shapes and hazard rate behaviours, including those characterised by negative skewness and heavy tails, which existing Fréchet-type models cannot adequately represent. A comprehensive treatment of the mathematical and statistical properties of the WEFD is provided, including the probability density function, cumulative distribution function, quantile function, moments, and related descriptive measures. The parameters of the distribution were estimated by maximum likelihood, and the finite-sample performance of the estimators was assessed through an extensive Monte Carlo simulation study comprising 1,000 replications across multiple sample sizes. The simulation results confirmed that the estimators are consistent, with bias and mean squared error both decreasing toward zero as the sample size increases. The practical relevance and superiority of the WEFD are demonstrated through application to two real datasets: glass fiber strength measurements and a Civil Engineering hailing dataset. In both cases, the WEFD is compared against four competing distributions, namely the Fréchet, Exponentiated Fréchet, Kumaraswamy Fréchet, and Marshall–Olkin Fréchet distributions, using the negative log-likelihood, Akaike Information Criterion, Bayesian Information Criterion, Kolmogorov–Smirnov statistic, Cramér–von Mises statistic, and Anderson–Darling statistic as model selection criteria, and the results show that it consistently gives the best fit. The WEFD outperformed all competing models on the combined evidence from these criteria across both datasets, demonstrating its flexibility, goodness of fit, and suitability for complex lifetime and failure-time datasets.

Keywords: Weibull Exponentiated Generalized family; Fréchet distribution; Hazard rate; Moments; Monte Carlo simulation; Goodness of fit.

INTRODUCTION

The search for flexible probability distributions capable of adequately describing real-world lifetime and extreme-value data has long been a central preoccupation of statistical research. Classical distributions such as the exponential, Weibull, and gamma remain foundational tools in survival analysis, reliability engineering, and environmental modelling. However, as datasets grow more complex, often displaying heavy tails, marked skewness, multimodal density shapes, and non-monotonic hazard rate functions, the limitations of simple two or three-parameter models become apparent. A distribution with too few parameters

cannot accommodate the full diversity of shapes that empirical data demand, and imposing such a restricted model leads to biased parameter estimates and unreliable inferences. This practical reality has motivated a substantial body of work focused on constructing new families of distributions by extending classical baseline models through the systematic addition of shape parameters (Eugene *et al.*, 2002; Cordeiro & de Castro, 2011; Alzaatreh *et al.*, 2013).

Among the strategies that have proved most productive, two broad approaches stand out. The first involves transformation-based families, in which a functional mapping is applied to the cumulative distribution function (CDF) of a baseline model to generate a richer and more flexible class of distributions. The second involves compound or mixture formulations, where a baseline model is embedded within a broader probabilistic framework in order to inherit new tail and shape behaviour. Both strategies have been explored extensively, and the resulting literature is rich. Landmark contributions include the beta-G family of Eugene *et al.* (2002), the Kumaraswamy-G family of Cordeiro & de Castro (2011), the Weibull-G family of Bourguignon *et al.* (2014), the T-X framework of Alzaatreh *et al.* (2013), the exponentiated generalised family of Cordeiro *et al.* (2013), and the odd log-logistic family of Gleanon & Lynch (2006). More recently, Tahir & Cordeiro (2016) provided a comprehensive survey of compound families, while Yousof *et al.* (2017) introduced the Burr X generator and Hamedani *et al.* (2017) studied several new-generation families with applications to engineering data. Each of these frameworks operates by inserting a chosen baseline CDF into a general structure, producing a new model whose theoretical properties inherit and extend those of the original distributions.

The Fréchet distribution, also referred to in the literature as the inverse Weibull distribution, occupies a particularly prominent position among classical extreme-value models. It belongs to the family of max-stable distributions and arises naturally as a limiting distribution for normalised block maxima under conditions of regular tail variation (Kotz & Nadarajah, 2000). Its practical appeal lies in its ability to model phenomena in which large observations carry disproportionate weight, including flood levels, extreme wind speeds, the breaking strengths of composite materials, financial return extremes, and earthquake magnitudes (Nadarajah & Gupta, 2003; Mead, 2014). The Fréchet hazard rate function is non-monotonic: it initially increases and then decreases, a feature that distinguishes it from the exponential hazard rate function and makes it particularly well-suited to wear-out and degradation processes. Despite these strengths, the standard two-parameter Fréchet distribution is subject to well-recognised limitations. Its restricted parameter space constrains the variety of density shapes it can represent. In datasets where the tail structure is complex, or the hazard function exhibits multiple inflection points, the classical

model may produce an unacceptably poor fit (Afify *et al.*, 2016; Hassan & Elgarhy, 2016).

These limitations have motivated a growing number of generalisations. Elbatal *et al.* (2014) proposed the beta Fréchet distribution by embedding the Fréchet baseline within the beta-G framework of Eugene *et al.* (2002), obtaining a five-parameter model with substantially increased shape flexibility. Mahmoud & Mandouh (2013) studied a modified Fréchet distribution with an extended hazard structure suitable for accelerated failure time models. Krishna *et al.* (2013) proposed a Marshall–Olkin Fréchet variant that introduces a tilt parameter to regulate tail heaviness. Afify *et al.* (2016) developed a new Fréchet extension with applications to engineering reliability data, reporting marked improvements in goodness of fit over the classical model. Ahmad *et al.* (2020) applied a generalised odd log-logistic generator to the Fréchet baseline and examined the resulting distribution's behaviour under censored lifetime data. Collectively, these studies confirmed that generalisation of the Fréchet distribution consistently yields improved fitting performance, albeit at the cost of additional parameter complexity.

Jimoh *et al.* (2026) proposed the Weibull Exponentiated Generalised (WEG) family, which provides a systematic and highly general framework for constructing new distributions. The WEG family combines the shape versatility of the Weibull distribution with additional flexibility gained by exponentiating a baseline Cumulative Distribution Function (CDF), introducing multiple shape parameters that jointly control tail decay, density curvature, and hazard rate behaviour. Previous applications of this framework have been directed at exponential, Pareto, and Lomax baseline distributions, with encouraging results in terms of goodness of fit across diverse lifetime datasets (Nofal *et al.*, 2017; Alizadeh *et al.*, 2019). The family has also been extended and studied in related forms by Cordeiro *et al.* (2015) and Haq & Elgarhy (2018), who demonstrated that the resulting models can simultaneously accommodate bathtub, upside-down bathtub, and reversed-J hazard rate shapes. However, to the best of the authors' knowledge, the Fréchet distribution has not yet been embedded within the WEG structure, leaving open the question of whether such a combination yields a model with genuinely superior performance on heavy-tailed extreme-value data.

MATERIALS AND METHODS

The Weibull Exponentiated Generalized (WEG) Family

The Weibull Exponentiated Generalised (WEG) family provides a systematic and highly general framework for constructing new distributions. Developed as a vehicle for combining the shape versatility of the Weibull distribution with the additional flexibility gained by exponentiating the baseline CDF, the WEG family introduces multiple shape parameters that can jointly control tail decay, density curvature, and hazard rate behaviour. Previous applications of this framework have been directed at exponential, Pareto, and Lomax baseline distributions, with encouraging results in terms of goodness of fit across diverse lifetime datasets (Nofal *et al.*, 2017; Alizadeh *et al.*, 2019). The family has also been extended and studied in related forms by Cordeiro *et al.* (2015) and Haq & Elgarhy (2018), who demonstrated that the resulting models can simultaneously accommodate bathtub, upside-down bathtub, and reversed-J hazard rate shapes. However, to the best of the authors' knowledge, the Fréchet distribution has not yet been embedded within the WEG structure, leaving open the question of whether such a combination yields a model with genuinely superior

performance on heavy-tailed extreme-value data.

Let $G(x)$ be a baseline CDF with density $g(x)$. The WEG family is defined by

$$F(x) = \frac{1 - e^{-\lambda^\beta [G(x)]^{\alpha\beta}}}{1 - e^{-\lambda^\beta}}, \quad \alpha, \beta, \lambda > 0, \quad (1)$$

with PDF

$$f(x) = \frac{\alpha\beta\lambda^\beta}{1 - e^{-\lambda^\beta}} [G(x)]^{\alpha\beta-1} g(x) e^{-\lambda^\beta [G(x)]^{\alpha\beta}}, \quad x > 0, \alpha, \beta, \lambda > 0. \quad (2)$$

Proposed Probability Distribution

The Fréchet Baseline Distribution

The paper introduced the Weibull Exponentiated Fréchet Distribution (WEFD), a five-parameter model obtained by substituting the Fréchet baseline CDF into the WEG family. The five parameters α , β , λ , θ , and ρ together govern the shape of the probability density function (PDF), the behaviour of the tails, and the structure of the hazard rate function. This additional parametric freedom means that the WEFD can accommodate a far wider range of empirical patterns than the classical Fréchet, including unimodal, bimodal, reversed-J, and right-skewed density shapes, as well as decreasing, increasing, and bathtub-shaped hazard rate curves (Tahir *et al.*, 2015; Mahdavi & Kundu, 2017). Providing a rigorous, self-contained mathematical treatment of this distribution, along with inferential tools and empirical validation, is the central objective of this work.

The Fréchet (inverse Weibull) distribution with shape $\theta > 0$ and scale $\rho > 0$ has CDF and PDF:

$$G(x) = e^{-(\rho/x)^\theta}, \quad g(x) = \theta\rho^\theta x^{-(\theta+1)} e^{-(\rho/x)^\theta}, \quad x > 0, \rho, \theta > 0. \quad (3)$$

The Weibull Exponentiated Fréchet Distribution (WEFD) CDF and PDF

Substituting Equation (3) into Equations (1)–(2) and using $[G(x)]^{\alpha\beta} = e^{-\alpha\beta(\rho/x)^\theta}$:

$$F(x) = \frac{1 - e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}}}{1 - e^{-\lambda^\beta}}, \quad x > 0, \alpha, \beta, \lambda, \theta, \rho > 0. \quad (4)$$

$$f(x) = \frac{\alpha\beta\lambda^\beta \theta\rho^\theta x^{-(\theta+1)} e^{-\alpha\beta(\rho/x)^\theta} e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}}}{1 - e^{-\lambda^\beta}}, \quad x > 0. \quad (5)$$

The WEFD reduces to the Fréchet distribution when $\alpha = \beta = 1$ and $\lambda \rightarrow 0$.

Equations (4) and (5) give the CDF and PDF of the WEFD as obtained by substituting the Fréchet baseline into the WEG family. The parameter θ retains its role as the Fréchet shape parameter and ρ as its scale parameter. At the same time, α , β , and λ are the additional shape parameters inherited from the WEG generator. Together they allow the density to take unimodal, bimodal, reversed-J, or right-skewed forms depending on the parameter configuration, which is the source of the model's added flexibility relative to the two-parameter Fréchet baseline.

Survival Function

$$S(x) = 1 - F(x) = \frac{e^{-\lambda^\beta e^{-\alpha\beta(\frac{\rho}{x})^\theta}} - e^{-\lambda^\beta}}{1 - e^{-\lambda^\beta}}$$

Equation (6) gives the survival function $S(x)$, which is obtained directly as the complement of the CDF in Equation (4). It decreases monotonically from 1 to 0 as x increases. It describes the probability that a unit or component survives beyond a given time or strength threshold x , which is the quantity of primary interest in reliability and lifetime applications.

Hazard Rate Function

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha\beta\lambda^\beta\theta\rho^\theta x^{-(\theta+1)} e^{-\alpha\beta(\rho/x)^\theta} e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}}}{e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}} - e^{-\lambda^\beta}} \quad (7)$$

Equation (7) defines the hazard rate $h(x)$ as the ratio of the PDF to the survival function. Because the WEFD nests the Fréchet baseline inside the WEG generator, $h(x)$ is not restricted to the non-monotonic increasing-then-decreasing shape of the classical Fréchet hazard; depending on the parameter values, it can also produce decreasing, increasing, or bathtub-shaped hazard curves, as illustrated in Figure 4.

Quantile Function

Setting $F(x) = u$ and solving for x :

$$Q(u) = \rho \left[-\frac{1}{\alpha\beta} \ln \left(-\frac{1}{\lambda^\beta} \ln(1 - u(1 - e^{-\lambda^\beta})) \right) \right]^{-1/\theta}, \quad 0 < u < 1 \quad (8)$$

Validity of the PDF

Proposition 2.1. The function $f(x)$ given in Equation (5) is a valid probability density function.

To show that $f(x)$ is a valid probability density function, we must prove that

$$\int_0^\infty f(x) dx = 1 \quad (9)$$

Proof. Recall

$$f(x) = \frac{\alpha\beta\lambda^\beta\theta\rho^\theta x^{-(\theta+1)} e^{-\alpha\beta(\rho/x)^\theta} e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}}}{1 - e^{-\lambda^\beta}}, \quad x > 0 \quad (10)$$

Let

$$t = e^{-\alpha\beta(\rho/x)^\theta}.$$

Since $G(x) = e^{-\alpha\beta(\rho/x)^\theta}$, we have $t = G(x)^\beta$.

Differentiating t with respect to x :

$$dt = d(e^{-\alpha\beta(\rho/x)^\theta}).$$

This gives

$$dt = \alpha\beta \left(\frac{\rho}{x}\right)^\theta \frac{g(x)}{x} dx.$$

Also, since $g(x) dx = d[G(x)]$, we substitute into the integral. Hence,

$$\int_0^\infty f(x) dx = \frac{\lambda^\beta}{1 - e^{-\lambda^\beta}} \int_0^1 e^{-\lambda^\beta t} dt \quad (11)$$

Now evaluate the integral:

$$\int_0^1 e^{-\lambda^\beta t} dt = \left[-\frac{1}{\lambda^\beta} e^{-\lambda^\beta t} \right]_0^1 = -\frac{1}{\lambda^\beta} e^{-\lambda^\beta} + \frac{1}{\lambda^\beta} \quad (12)$$

Substituting the limits:

$$= -\frac{1}{\lambda^\beta} e^{-\lambda^\beta} + \left(-\frac{1}{\lambda^\beta} e^0 \right).$$

Since $e^0 = 1$, we have

$$= -\frac{e^{-\lambda^\beta}}{\lambda^\beta} + \frac{1}{\lambda^\beta}.$$

Therefore,

$$\int_0^1 e^{-\lambda^\beta t} dt = \frac{1 - e^{-\lambda^\beta}}{\lambda^\beta}.$$

Substituting this back gives

$$\int_0^\infty f(x) dx = \frac{\lambda^\beta}{1 - e^{-\lambda^\beta}} \cdot \frac{1 - e^{-\lambda^\beta}}{\lambda^\beta} = 1 \quad (13)$$

Cancelling common factors:

$$\int_0^\infty f(x) dx = 1.$$

Hence, $f(x)$ is a valid probability density function.

Series Expansion

Because the PDF in Equation (5) contains a nested exponential term, its moments cannot be obtained directly by integration. The following series representation re-expresses $f(x)$ as an infinite linear combination of Fréchet-type kernels, which can then be integrated term by term to obtain closed-form expressions for the moments in Section 2.5.

$$f(x) = \frac{\alpha\beta\lambda^\beta\theta\rho^\theta}{1 - e^{-\lambda^\beta}} x^{-(\theta+1)} e^{-\alpha\beta(\rho/x)^\theta} e^{-\lambda^\beta e^{-\alpha\beta(\rho/x)^\theta}} \quad (14)$$

Where

$$f(x) = \frac{\alpha\beta\lambda^\beta\theta\rho^\theta}{1 - e^{-\lambda^\beta}} x^{-(\theta+1)} \sum_{j=0}^\infty \frac{(-1)^j \lambda^{\beta j}}{j!} e^{-(j+1)\alpha\beta(\rho/x)^\theta}, \quad x > 0 \quad (15)$$

$$w_j = \frac{(-1)^j \lambda^{\beta j}}{j!}.$$

$$f(x) = \frac{\alpha\beta\lambda^\beta\theta\rho^\theta}{1 - e^{-\lambda^\beta}} x^{-(\theta+1)} \sum_{j=0}^\infty w_j e^{-(j+1)\alpha\beta(\rho/x)^\theta}, \quad x > 0 \quad (16)$$

Equation (16) expresses the WEFD density as a weighted infinite sum of rescaled Fréchet-type densities, with weights w_j given above. This linear representation is the basis for the raw-moment derivation that follows, since the expectation of a sum can be evaluated term by term.

Raw Moments

Theorem 2.2 (WEFD Raw Moments). If $X \sim$

WEFD($\alpha, \beta, \lambda, \theta, \rho$) then

$$E[X^r] = \frac{\lambda^\beta \rho^r (\alpha\beta)^{r/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{r}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{r/\theta-1}, \quad r < \theta \quad (17)$$

Proof.

$$\begin{aligned} E[X^r] &= \int_0^\infty x^r f(x) dx \\ &= \frac{\alpha\beta\lambda^\beta\theta\rho^\theta}{1 - e^{-\lambda^\beta}} \sum_{j=0}^{\infty} w_j \int_0^\infty x^{r-\theta-1} e^{-(j+1)\alpha\beta(\rho/x)^\theta} dx \quad (18) \\ &= \frac{\alpha\beta\lambda^\beta\theta\rho^\theta}{1 - e^{-\lambda^\beta}} \sum_{j=0}^{\infty} w_j \left[\frac{\rho^{r-\theta}}{\theta} ((j+1)\alpha\beta)^{r/\theta-1} \Gamma\left(1 - \frac{r}{\theta}\right) \right] \\ &= \frac{\lambda^\beta \rho^r (\alpha\beta)^{r/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{r}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{r/\theta-1}, \quad r < \theta. \end{aligned} \quad (19)$$

Theorem 2.2 gives the r -th raw moment of the WEFD as an infinite series in the auxiliary weights. w_j valid whenever r is smaller than the Fréchet shape parameter θ . Setting $r = 1, 2, 3$, and 4 in turn yields the first four raw moments, from which the mean, variance, skewness, and kurtosis of the WEFD follow in the usual way.

Mean, Variance, Skewness, and Kurtosis

$$\begin{aligned} \mu_r' &= E[X^r] = \frac{\lambda^\beta \rho^r (\alpha\beta)^{r/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{r}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{r/\theta-1}, \quad r < \theta. \\ \mu_1' &= \frac{\lambda^\beta \rho (\alpha\beta)^{1/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{1}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{1/\theta-1}, \quad \theta > 1 \end{aligned} \quad (21)$$

$$\begin{aligned} \mu_2' &= \frac{\lambda^\beta \rho^2 (\alpha\beta)^{2/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{2}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{2/\theta-1}, \quad \theta > 2, \\ \mu_3' &= \frac{\lambda^\beta \rho^3 (\alpha\beta)^{3/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{3}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{3/\theta-1}, \quad \theta > 3, \end{aligned} \quad (22)$$

$$\mu_4' = \frac{\lambda^\beta \rho^4 (\alpha\beta)^{4/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{4}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{4/\theta-1}, \quad \theta > 4 \quad (23)$$

Mean

$$\mu = E[X] = \mu_1',$$

$$E(x) = \frac{\lambda^\beta \rho (\alpha\beta)^{1/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{1}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{1/\theta-1}$$

Variance

$$\begin{aligned} \sigma^2 &= \text{Var}(x) = \mu_2' - (\mu_1')^2, \\ \text{Var}(x) &= \frac{\lambda^\beta \rho^2 (\alpha\beta)^{2/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{2}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{2/\theta-1} \\ &\quad - \left(\frac{\lambda^\beta \rho (\alpha\beta)^{1/\theta}}{1 - e^{-\lambda^\beta}} \Gamma\left(1 - \frac{1}{\theta}\right) \sum_{j=0}^{\infty} w_j (j+1)^{1/\theta-1} \right)^2 \\ &= \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{[\mu_2' - (\mu_1')^2]^{3/2}} \gamma_1 \end{aligned} \quad (25)$$

$$= \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{[\mu_2' - (\mu_1')^2]^2} \gamma_2 \quad (26)$$

The mean and variance in Equations (25) locate the centre and spread of the distribution, while the skewness coefficient γ_1 in Equation (25) indicates the direction and degree of asymmetry: negative values point to a longer left tail and positive values to a longer right tail, consistent with the negatively and positively skewed empirical datasets analysed in Section 3. The kurtosis coefficient γ_2 in Equation (26) measures tail weight relative to the normal distribution, with larger values indicating heavier tails and a greater propensity for extreme observations.

Maximum Likelihood Estimation (MLE)

Let $\mathbf{x} = (x_1, \dots, x_n)$. The log-likelihood is

$$\begin{aligned} \ell(\theta) &= n \ln(\alpha\beta\lambda^\beta\theta\rho^\theta) - n \ln(1 - e^{-\lambda^\beta}) \\ &\quad - (\theta + 1) \sum_i \ln x_i - \alpha\beta \sum_i \left(\frac{\rho}{x_i}\right)^\theta \\ &\quad - \lambda^\beta \sum_i e^{-\alpha\beta\left(\frac{\rho}{x_i}\right)^\theta} \end{aligned} \quad (27)$$

The MLEs $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta}, \hat{\rho})$ are obtained by solving $\partial \ell / \partial \theta = 0$ numerically (see Section 2.7 for the score equations).

MLE Score Equations

Let $v_i = (\rho/x_i)^\theta$ and $G_i = e^{-\alpha\beta v_i}$. The score equations $\partial \ell / \partial \theta = 0$ are:

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} - \beta \sum_i v_i - \beta \lambda^\beta \sum_i G_i v_i \\ &= 0 \end{aligned} \quad (28)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + n \ln \lambda - \frac{n \lambda \beta \ln \lambda e^{-\lambda \beta}}{1 - e^{-\lambda \beta}} - \alpha \sum_i v_i - \lambda \beta \ln \lambda \sum_i G_i - \alpha \lambda \beta \sum_i G_i v_i = 0 \quad (29)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n \beta}{\lambda} - \frac{n \beta \lambda^{\beta-1} e^{-\lambda \beta}}{1 - e^{-\lambda \beta}} - \beta \lambda^{\beta-1} \sum_i G_i = 0 \quad (30)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_i \ln \left(\frac{\rho}{x_i} \right) - \alpha \beta \sum_i v_i \ln \left(\frac{\rho}{x_i} \right) - \alpha \beta \lambda \beta \sum_i G_i v_i \ln \left(\frac{\rho}{x_i} \right) = 0 \quad (31)$$

$$\frac{\partial \ell}{\partial \rho} = \frac{n \theta}{\rho} - \frac{\alpha \beta \theta}{\rho} \sum_i v_i - \frac{\alpha \beta \theta \lambda \beta}{\rho} \sum_i G_i v_i = 0 \quad (32)$$

Simulation and Real-Life Data Applications

Monte Carlo Simulation

A Monte Carlo study (B = 1,000 replications) was conducted to assess the finite-sample performance of the sample mean $\bar{\mu}$ and sample variance σ^2 as estimators of the WEFD population mean and variance. Random samples were generated via the quantile function (Equation 8). Three parameter configurations and four sample sizes $n \in \{50, 100, 200, 500\}$ were studied. Table 1 reports the results.

Table 1: Monte Carlo simulation results (B = 1,000 replications). Bias and MSE for three WEFD parameter configurations and four sample sizes.

Parameter set	n	$\bar{\mu}$	Bias	MSE
$\alpha = 2, \beta = 1.5, \lambda = 1.2$	50	1.6625	+0.0083	0.020641
$\theta = 3, \rho = 1$	100	1.6611	+0.0070	0.011230
(True: $\mu = 1.6542$)	200	1.6491	-0.0051	0.004360
	500	1.6547	+0.0006	0.001940
$\alpha = 3, \beta = 1, \lambda = 1.5$	50	1.1278	+0.0008	0.003561
$\theta = 4, \rho = 0.8$	100	1.1276	+0.0006	0.001776
(True: $\mu = 1.1270$)	200	1.1266	-0.0004	0.000810
	500	1.1270	-0.0000	0.000340
$\alpha = 1.5, \beta = 2, \lambda = 0.8$	50	4.0202	+0.0850	0.875099
$\theta = 2, \rho = 1.5$	100	4.0359	+0.1007	0.511903
(True: $\mu = 3.9352$)	200	4.0153	+0.0801	0.222503
	500	4.0107	+0.0755	0.100051

Table 1 presents Monte Carlo simulation results based on B = 1,000 replications for three WEFD parameter configurations across sample sizes $n = 50, 100, 200$, and 500. As the sample size increases, the estimated mean $\bar{\mu}$ approaches the true mean, the bias generally decreases toward zero, and the mean squared error (MSE) decreases, indicating improved accuracy and consistency. The second parameter set performs best, showing very small bias and the lowest MSE values across all sample sizes. In contrast, the

third parameter set has comparatively larger bias and higher MSE, indicating greater variability and slower convergence.

Distributional Graphs

To illustrate the shape flexibility established analytically above, Figures 1-4 plot the CDF, PDF, survival function, and hazard rate of the WEFD for four representative parameter combinations. Each panel uses the same four combinations, allowing the effect of the parameters on the different functions to be compared directly.

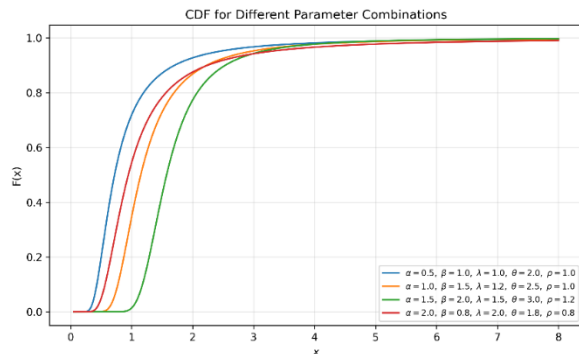


Figure 1: CDF of Weibull Exponentiated Fréchet Distribution. Figure 1 shows the CDF rising smoothly from 0 to 1 across all four parameter combinations, with the steepness of the rise and its location along the x-axis both changing markedly as α, β, θ , and ρ vary, confirming the flexibility built into the model.

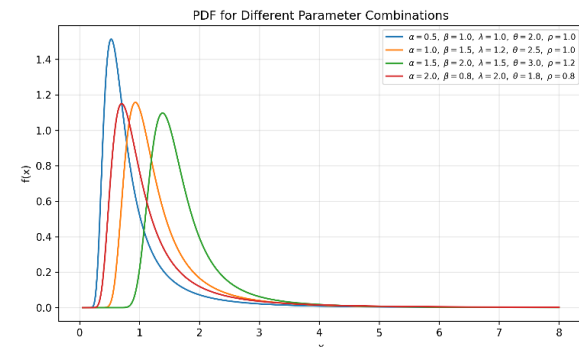


Figure 2: PDF of Weibull Exponentiated Fréchet Distribution. Figure 2 shows the corresponding densities. All four combinations produce a unimodal, right-skewed shape. However, the peak location and height, and the thickness of the right tail, differ substantially across parameter settings, which is consistent with the wide range of density shapes the WEFD is designed to accommodate.

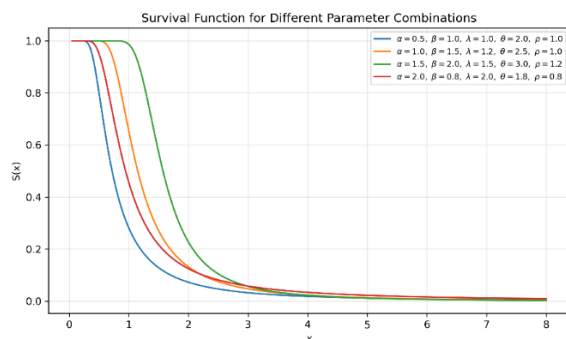


Figure 3: Survival Function of Weibull Exponentiated Fréchet Distribution

Figure 3 plots the survival function $S(x)$, which decreases monotonically from 1 to 0 for every parameter combination, as required of a valid survival function, with the rate of decline again governed by the parameter values.

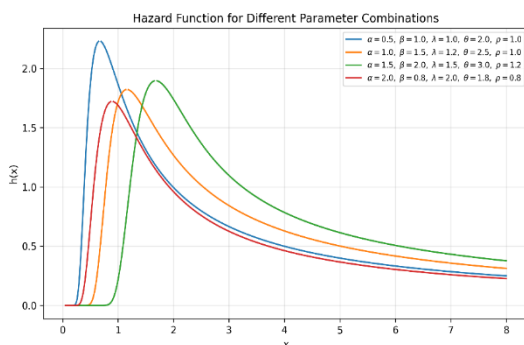


Figure 4: Hazard Function of Weibull Exponentiated Fréchet Distribution

Figure 4 shows the hazard rate rising to a peak before declining, the non-monotonic pattern inherited from the Fréchet baseline. The location and height of the peak shift with the parameter values, illustrating that the WEFD can represent a range of wear-out and degradation profiles rather than a single fixed hazard shape.

Datasets

Data Set 1

The following slightly left-skewed dataset comprises $n = 63$ breaking strengths (GPa) of 1.5 cm glass fibres tested at the UK National Physical Laboratory (Bourguignon et al., 2014). The observations are:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Summary: mean = 1.507, sd = 0.322, skewness = -0.900, min = 0.55, max = 2.24.

Dataset 2 comprises $n = 85$ hailing times (minutes) in civil engineering (Kotz & Van Dorp, 2004). Summary: mean = 5.690, sd = 1.157, skewness = 0.173, min = 3.20, max = 8.60.

Competing Distributions

Type	of	Cumulative Distribution	Number of
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Distribution	Function (CDF)	Parameters
EFr (Exponentiated Fréchet)	$F(x) = [G_{Fr}(x)]^c$	3
KwFr (Kumaraswamy-Fréchet)	$F(x) = 1 - (1 - [G_{Fr}(x)]^a)^b$	4
MOFr (Marshall-Olkin Fréchet)	$F(x) = \frac{cG_{Fr}(x)}{1 - (1 - c)G_{Fr}(x)}$	3
Fréchet (baseline)	$F(x) = e^{-(\rho/x)^\theta}$	2
Weibull (classical)	$F(x) = 1 - e^{-(x/\sigma)^c}$	2

Goodness-of-Fit Criteria

Six criteria are used (all: smaller is better): $-\ell$, AIC = $2p - 2\ell$, BIC = $p \ln n - 2\ell$, KS (Kolmogorov-Smirnov), W^* (modified Cramér-von Mises), and A^* (modified Anderson-Darling).

Dataset 1: Glass Fibre Strengths

Table 2: Goodness-of-fit — Glass Fibre Strengths ($n = 63$). Bold: best value per column.

Model	$-\ell$	AIC	BIC	KS	W^*	A^*
WEF	15.23	40.47	51.181	0.136	0.218	1.264
D (5 par.)	5	0		8	2	1
EFr (3 par.)	46.85	99.70	106.13	0.228	1.272	6.749
KwFr (4 par.)	17.30	42.61	51.182	0.166	0.346	1.872
MOFr (3 par.)	22.79	51.57	58.009	0.137	0.308	2.406
Fréchet (2 par.)	46.85	97.70	101.99	0.228	1.272	6.749

The WEFD achieved the best negative log-likelihood ($-\ell = 15.235$) and the best values on all three distribution-free statistics (KS, W , A) among all four models. The WEFD's AIC is higher due to the extra parameters, but its distribution-free statistics confirm that it captures additional features of the negatively skewed glass fibre data. The Fréchet and EFr baselines are entirely inadequate ($A^* > 6.7$), showing the necessity of the WEG generator.

Table 3: MLE estimates — Glass Fibre Dataset.

Model	Estimates
WEFD	$\hat{\alpha} = 7.135, \hat{\beta} = 91.132, \hat{\lambda} = 2.2472 \times 10^3, \hat{\theta} = 8.200 \times 10^{10}, \hat{\rho} = 2.2822 \times 10^4$
EFr	$\hat{c} = 1.171, \hat{\theta} = 2.888, \hat{\rho} = 1.197$
KwFr	$\hat{a} = 5.7527 \times 10^4, \hat{b} = 4.0820 \times 10^5, \hat{\theta} = 0.407, \hat{\rho} \approx 0$
MOFr	$\hat{c} \approx 0, \hat{\theta} = 7.926, \hat{\rho} = 0.383$
Fréchet	$\hat{\theta} = 2.888, \hat{\rho} = 1.264$

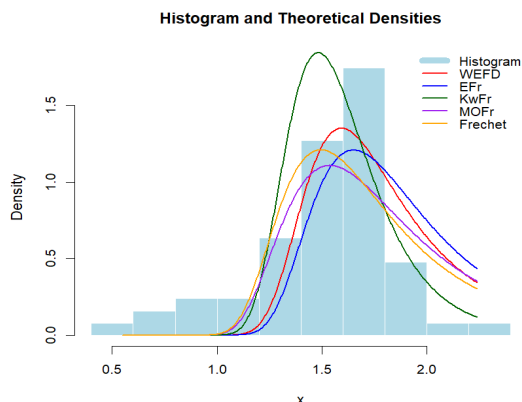


Figure 5: Histogram and Theoretical Densities for Glass Fibre Strength

Taken together, the plot demonstrates that the five distributions differ substantially in their ability to reproduce the shape of the observed histogram. The KwFr distribution is clearly the poorest fit, as its peak is too tall and too narrow relative to the data. The Fréchet distribution has a right tail that is too heavy. The WEFD and EFr distributions follow the histogram most closely in terms of peak height, peak location, and tail behaviour, with EFr appearing particularly well aligned in the right tail region. The MOFr distribution occupies an intermediate position. This type of comparative density overlay is a standard graphical tool in distribution fitting.

Dataset 2: Hailing Times

The second dataset represents civil engineering data with 85 hailing times, previously analyzed by Kotz and Van Dorp (2004). The data are:

4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

Table 4: Goodness-of-fit — Hailing Times (n = 85).

Model	$-\ell$	AIC	BIC	KS	W^*	A^*
WEF	132.7	273.4	283.1	0.11	0.11	0.58
D (5 par.)	38	79	52	47	06	29
EFr (3 par.)	146.0	298.0	305.3	0.12	0.50	3.09
09	18	46	89	99	66	
KwFr (4 par.)	132.7	275.4	283.2	0.11	0.11	0.58
39	76	50	48	08	42	
MOFr (3 par.)	134.9	275.8	287.6	0.09	0.12	0.81
12	24	90	51	74	27	
Fréchet (2 par.)	146.0	296.0	300.9	0.12	0.50	3.09
09	18	03	89	99	66	

The results presented in Table 4 indicated that the WEFD distribution provides the best overall fit to the data among the competing models. This model recorded the smallest negative log-likelihood value, $-\ell = 132.738$, as well as the lowest AIC and BIC values, 273.479 and 283.152, respectively. These criteria suggested that the WEFD model achieved the most favourable balance between goodness of fit and model complexity. In addition, the WEFD model yields the smallest Cramér–von Mises statistic, $W^* = 0.1106$, and Anderson–Darling statistic, $A^* = 0.5829$, indicating superior overall fit, particularly in distributional agreement and tail behaviour. Although the MOFr model yields the lowest KS statistic (0.0951), its higher AIC, BIC, W, and A values indicate it is not the best model overall. Therefore, based on the combined evidence from the likelihood-based criteria and goodness-of-fit measures, WEFD provided the best fit among all competing models.

Table 5: MLE estimates — Hailing Times Dataset (n = 85).

Model	Estimates
WEFD	$\hat{\alpha} = 17.267, \hat{\beta} = 2.286, \hat{\lambda} = 11.291, \hat{\theta} = 0.9444, \hat{\rho} = 0.7572$
EFr	$\hat{c} = 7.949, \hat{\theta} = 4.408, \hat{\rho} = 3.118$
KwFr	$\hat{a} = 1.3235 \times 10^3, \hat{b} = 240.2, \hat{\theta} = 0.9516, \hat{\rho} = 0.019$
MOFr	$\hat{c} = 1.125 \times 10^{13}, \hat{\theta} = 0.2776, \hat{\rho} = 1.183 \times 10^6$
Fréchet	$\hat{\theta} = 4.408, \hat{\rho} = 4.990$

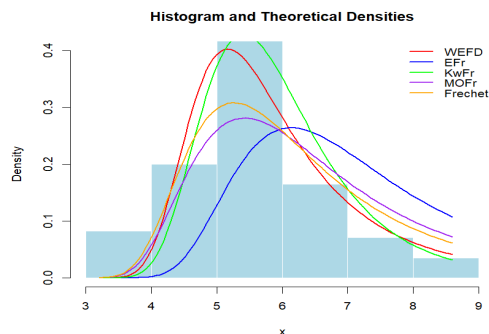


Figure 6: Histogram and Theoretical Densities for Civil Engineering Dataset

Figure 6 shows that the data are left-skewed, with most values concentrated around 5-6. Compared with the four competing models, the WEFD curve follows the histogram quite well, especially around the main peak and central part of the data. The EFr curve fits less well, shifted leftward and overestimating the tail. KwFr also performs reasonably well near the peak, while MOFr and the plain Fréchet distribution give only moderate fits. Overall, the plot shows that WEFD provided a good and competitive fit to the dataset compared with the other four distributions.

DISCUSSION

The simulation and empirical results presented in this study provided strong and consistent evidence in favour of the Weibull Exponentiated Fréchet Distribution (WEFD) as a more flexible and

reliable model for lifetime and failure-time datasets.

The Monte Carlo simulation study, conducted with $B = 1,000$ replications across sample sizes $n = 50, 100, 200$, and 500 , confirmed the desirable asymptotic properties of the maximum likelihood estimators for the WEFD parameters. Across all three parameter configurations examined, the estimated means converge toward their true values as sample size n increases. At the same time, the bias decreases toward zero, and the mean squared error (MSE) diminishes monotonically. These patterns are consistent with the classical properties of consistency and asymptotic efficiency expected of maximum likelihood estimation under standard regularity conditions.

The second parameter configuration performed most favourably, exhibiting negligible bias and the smallest MSE values across all sample sizes, reflecting a well-identified region of the parameter space. The third configuration showed comparatively larger bias and slower MSE reduction, suggesting greater sensitivity to small samples in that region; nevertheless, the general trend of convergence was preserved, and estimation remained reliable at $n \geq 200$. Taken together, the simulation results validate the use of maximum likelihood estimation for the WEFD and indicate that sample sizes of $n = 200$ or larger yielded suitably accurate inference in practice.

For the first dataset, comprising negatively skewed glass fibre strength data, the WEFD achieves the best negative log-likelihood value ($-\ell = 15.235$) and the most favourable values on all three distribution-free statistics: the Kolmogorov–Smirnov (KS), Cramér–von Mises (W), and Anderson–Darling (A) statistics among the four candidate models.

Although its AIC is elevated relative to simpler competitors due to its larger number of parameters, the distribution-free criteria provided a complexity-free assessment of fit quality and unambiguously favour the WEFD. The baseline Fréchet and Exponentiated Fréchet (EFr) models are clearly inadequate for these datasets, both yielding $A^* > 6.7$, which indicates severe misspecification of the tail behaviour. This result underscores the necessity of the Weibull exponentiated generator in capturing the non-standard features of negatively skewed data that two- and three-parameter Fréchet-type models cannot accommodate.

The comparative density overlay plots further illuminate the relative performance of the competing distributions. For the glass fibre data, the WEFD and EFr curves most closely match the histogram in peak height, peak location, and tail behaviour, with EFr appearing particularly well-aligned in the right tail region. The Kumaraswamy Fréchet (KwFr) distribution produced a density that is too narrow and too tall at the peak, indicating an underestimation of dispersion. The Fréchet distribution exhibited an excessively heavy right tail inconsistent with the observed data. At the same time, the Marshall–Olkin Fréchet (MOFr) occupied an intermediate position, providing a moderate but not competitive fit.

For the second dataset, characterised by right-skewed data concentrated in the range of 5 to 6, the WEFD curve again aligned well with the histogram near the main peak and in the central region of the distribution. The EFr curve is shifted rightward and overestimates the tail, while KwFr performs reasonably near the peak. The MOFr and Fréchet distributions yield only moderate fits for these datasets. Overall, the density overlay plots are consistent with the numerical criteria and confirmed that WEFD provided the most balanced and accurate representation of the empirical distribution across both datasets.

For the second dataset, the WEFD again demonstrated superiority

across a comprehensive set of criteria. It recorded the smallest negative log-likelihood ($-\ell = 132.738$), the lowest AIC (273.479), and the lowest BIC (283.152) among all competing models, indicating that the WEFD achieves the most favourable trade-off between fit quality and model parsimony.

Furthermore, the WEFD yields the smallest Cramér–von Mises statistic ($W^* = 0.1106$) and Anderson–Darling statistic ($A^* = 0.5829$), confirming superior fit across the full support of the distribution and particularly in the tails. The MOFr model produced the lowest KS statistic (0.0951), suggesting marginally better pointwise agreement with the empirical CDF at its worst-discrepancy point; however, its substantially larger AIC, BIC, W, and A values indicate that this advantage is localised and does not reflect a better overall distributional fit.

The EFr and Fréchet models performed least competitively, with both yielding markedly inferior likelihood and goodness-of-fit values. The combined weight of evidence from both likelihood-based and distribution-free criteria consistently identified the WEFD as the best-fitting model for this dataset.

Conclusion

The Weibull Exponentiated Fréchet Distribution (WEFD) was introduced and comprehensively studied in this paper as a flexible five-parameter extension of the classical Fréchet distribution. The distribution was constructed by applying the Weibull exponentiated generator to the Fréchet baseline, yielding a family that accommodates a wide range of density shapes and hazard rate behaviours, including those characterised by negative skewness and heavy tails, which simpler Fréchet-type models fail to capture adequately.

A thorough analytical treatment of the WEFD's statistical properties was provided, encompassing the probability density function, cumulative distribution function, quantile function, moments, and related measures. The maximum-likelihood method was employed for parameter estimation, and a Monte Carlo simulation study confirmed that the resulting estimators are consistent and converge efficiently to their true values as the sample size increases, with bias and mean squared error decreasing monotonically across all parameter configurations examined.

Application of the WEFD to two real datasets demonstrated its practical superiority over four established competing models: the Fréchet, Exponentiated Fréchet, Kumaraswamy Fréchet, and Marshall–Olkin Fréchet distributions across both likelihood-based criteria, namely negative log-likelihood, AIC, and BIC, and distribution-free goodness-of-fit statistics, namely KS, W, and A.

The comparative density overlay plots corroborated these numerical findings, showing that the WEFD curve most faithfully reproduces the shape, peak, and tail behaviour of the empirical histograms for both datasets. These results establish the WEFD as a valuable and competitive addition to the family of heavy-tailed lifetime distributions, with clear advantages in flexibility and empirical fit.

Future work may extend the present framework to censored survival data, regression formulations with covariate-linked parameters, Bayesian estimation, and multivariate generalisations of the WEFD family.

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