

UNIFICATION OF MULTIGROUPS AND MULTISSET GROUPS

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ABSTRACT

This paper presents a unified approach to studying multigroup and multiset group by examining their structural relationships within a common algebraic framework. The work investigates the connection between these two generalized group concepts and establishes that every multigroup is a multiset group, while the converse does not necessarily hold. This observation motivates a new paradigm at the intersection of multigroup and multiset group theory: the multigroup set. The proposed approach provides a broader multiset context for understanding group-like structures and offers a unified perspective that captures the essential properties shared by both concepts. This study contributes to the ongoing development of multiset-based algebraic structures and opens new directions for further research.

Keywords: Multiset, Multiset point, Multigroup, Multiset Group, Multigroup set

INTRODUCTION

In abstract algebra, groups play an important role in understanding symmetry and structure. Over time, researchers have sought to generalize this idea to develop larger, more complex systems in which a single operation can produce multiple outcomes. This motivation led to the development of multigroups, an algebraic structure that extends classical group theory by allowing the binary operation to be multivalued rather than a single-valued function. This whole idea was based on multisets.

The concept of multisets (msets for short) is a generalization of a classical set. In Cantor's sets, objects are not allowed to repeat; when repetition is allowed, a mathematical structure known as an mset emerges. An mset is an unordered collection of objects in which objects are allowed to repeat. The occurrence of an individual object in the mset is called its element, and each indistinguishable element might have a finite number in the mset. The number of occurrences of an individual element in a mset is called the multiplicity of the element, which is finite in most applications and contributes to the cardinality of the mset, as discussed by Blizard (1989) and Ibrahim et al. (2011).

Nazmul et al. (2013) generalized the concept of a group to a multigroup with respect to multisets and studied some of its properties. An algebraic structure of a multiset whose underlying set is a group is a multigroup if some set of axioms are satisfied. It was shown that the intersection of two multigroups is a multigroup, but the union is not. This algebraic structure extends or generalizes a group because it is defined with reference to multisets, incorporates multiplicities as count functions, and follows the trend of other non-classical groups.

Tella and Daniel (2013a) attempted to study classical group theory in the context of multisets, and they stated that an algebraic structure is a multigroup if and only if its root set is a group. However, this is not always true because the root set could be a group, but the multiset defined over the root set is not always a

multigroup. However, this definition of multigroup does not coincide with other non-classical groups such as fuzzy group, fuzzy multigroup, intuitionistic fuzzy group, among others.

Afterwards, Tella and Daniel (2013b) make another attempt to define a symmetric group in the multiset context and derived the analogue of Cayley's theorem.

Awolola and Ibrahim (2016) studied multigroups via multisets, presented some properties of multigroups, and investigated analogues of isomorphism theorems.

Awolola and Ejegwa (2017) proposed the idea of the order of an element with respect to a multigroup as a continuation of the study of various algebraic structures of multisets and outlined some related results.

Tripathy et al. (2017) attempted to generalize the concept of a group in the multiset context, define a multiset subgroup and study some of its basic properties.

Das and Roy (2021) first introduced the notion of multi-points (mpoints, for short) and multi-real points (mrpoints, for short), and presented their definitions and some basic properties. However, despite its richness, this conceptual approach was limited to multi-metric spaces and related applications. Tella et al. (2024) reviewed operations on collections of multi-points of multisets, including addition, union, intersection, difference, symmetric difference, complement, arithmetic multiplication, raising to an arithmetic power, and scalar multiplication. Furthermore, some algebraic properties of these operations, such as commutativity, idempotency, associativity, distributivity and De Morgan's laws, are examined. The concept of a root set multi-point collection of a multi-point collection is examined under the union and intersection of such collections.

Unlike the generalization by Nazmul et al. (2013), where the multigroup was defined with reference to multisets, incorporating the notion of multiplicities as count functions, Tripathy et al. (2017) generalize the concept of group and define the multiset group using the concept of binary mset composition to enhance the count function of the multiplicities. In the former, the multigroup has some limitations, especially when relaxing the count function of a commutative multigroup; the root set is not commutative. However, for the latter, the relaxed root set is commutative.

This paper examines groups in a unified multiset context, highlighting both their theoretical significance and their growing relevance in modern mathematics. By re-examining multigroups through a modern multipoint perspective on multisets, this paper will uncover deeper connections with classical algebra while opening new avenues for research.

MATERIAL AND METHODS

Msets

Definition 1: (mset) (Debnath & Debnath, 2019)

An mset is a collection of elements that may repeat. Formally if S is a set of elements, an mset A drawn from the set S is represented by a function $C_A: S \rightarrow \mathbb{N}$, where $\mathbb{N}$ represents the set of non-

negative integers. For each $x \in S$, $C_A(x)$ is the characteristic value of x in A and indicates the number of occurrences of the element x in A . A mset A is a set if

$$C_A(x) \in \{0,1\} \forall x \in S.$$

The symbolic representation $x \in^k A$ means x belongs to A exactly k times.

An mset M drawn from the set $X = \{x_1, x_2, x_3, \dots, x_n\}$ can be represented as

$$M = \{x_1, x_2, x_3, \dots, x_n\}_{m_1, m_2, m_3, \dots, m_n}, \quad \text{or} \quad M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$$

or $\{m_1x_1, m_2x_2, \dots, m_nx_n\}$ or $\{x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}\}$ where $m_i = C_M(x_i)$, $i = 1, 2, \dots, n$, as it depends on one's taste of convenience.

However, for this paper, we shall resort to the use of $M_0 = \{(m_1)x_1, (m_2)x_2, \dots, (m_n)x_n\}$ where $M = [x_1, x_1, \dots, x_1(m_1 \text{ times}), x_2, x_2, \dots, x_2(m_2 \text{ times}), \dots, x_n, x_n, \dots, x_n(m_n \text{ times})]$. Each $(m_i)x_i$ is called an mset point and M_0 is a set of distinct mset points following the axiom of unique multiplicity (Blizard, 1989).

Note that for any mset M over X , $(m)x \in M_0 \Leftrightarrow m > 0$ and $(m)x \notin M_0 \Leftrightarrow m = 0$

However, $\forall x[(m)x \in M_0 \Rightarrow m = 0] \Rightarrow M_0 = \emptyset$

Example 1

Let $X = \{a, b, c, d, e\}$ and $M = [a, a, b, b, b, b, d, d, d, d, e]$ an mset drawn from X . Then $M_0 = \{(2)a, (4)b, (5)d, (1)e\}$

Definition 2: (Cardinality of a mset) (Debnath & Debnath, 2019).

The cardinality of a mset A drawn from a set S denoted by $\text{card } A$ is defined by

$\text{card } A = \sum_{x \in S} m|(m)x \in A_0$. It is also denoted by $|A|$. An mset A over the set S is said to be finite if $|A| < \infty$. We denote the class of all finite msets over S by $\mathfrak{M}(S)$.

Definition 3: (root set) (Debnath & Debnath, 2019)

Let $A \in \mathfrak{M}(S)$. Then the root set or support set of A denoted A^* is given by the expression

$$A^* = \{x \in S | (m)x \in A_0\}. \quad \text{For instance, for } A = [a, a, b, b, b, b, c, c, c],$$

then $A_0 = \{(2)a, (3)b, (4)c\}$ and $A^* = \{a, b, c\}$ since $2, 3, 4 > 0$. In particular, $x \in A^* \Leftrightarrow (m)x \in A_0$

Definition 4: (equality of mset points) (Tella, 2020)

Two mset points $(m)x$ and $(n)y$ are equal, denoted by $(m)x = (n)y$ if and only if $m = n$ and $x = y$

Proposition 1: The mset points equality relation is an equivalence relation

Proof:

Very clear from the definition 4 and the fact that the equality (identical) relation $=$ is also an equivalence relation

Note that $(m)x \neq (n)y := m \neq n \wedge x = y$, $(m)x \neq (n)y := m = n \wedge x \neq y$

and $(m)x \neq (n)y := m \neq n \wedge x \neq y$

Definition 5: (Equality of msets): Let $A, B \in \mathfrak{M}(S)$. Then the msets A and B are said to be m-equal, denoted $A =_m B$ if and only if $A_0 = B_0$. We denote the m-unequal by $A \neq_m B$ if and only

if $A_0 \neq B_0$

Definition 6: (submset) (Tella, 2020)

Let $A, B \in \mathfrak{M}(S)$. The mset A is a submultiset (submset for short) of B denoted by $A \subseteq B$ or $B \supseteq A$ if and only if for any $(m)x \in A_0$, $(n)x \in B_0$, we have $m \leq n \forall x \in S$.

If $A \subseteq B$ and $A \neq_m B$, then A is called the proper submset of B denoted $A \subset B$

Multigroups

Definition 7: (multigroup) Nazmul et al., (2013)

Let X be a group and $A \in \mathfrak{M}(X)$. Then the mset A_0 is called a multigroup (mgrou, for short) if it satisfies the following conditions:

i. for any $(m)x$ and $(n)y$ such that $m = C_A(x)$, $n = C_A(y)$ and $k = C_A(xy)$ then

$$k \geq \min\{m, n\} \forall x, y \in X$$

ii. for any $(m)x$ and $(n)x^{-1}$ such that $m = C_A(x)$ and $n = C_A(x^{-1})$, then $n \geq m \forall x \in X$

Note that for any $(m)x$ and $(n)x^{-1}$ such that $m = C_A(x)$ and $n = C_A(x^{-1})$, then

$n = m \forall x \in X$ and for any $(m)x$ and $(n_e)e$ such that $m = C_A(x)$, $n_e = C_A(e)$, then

$n_e \geq m \forall x \in X$ where $e \in X$ is the identity. Let also $(k)x$ and $(m)x^n$ be such that $k = C_A(x)$ and $m = C_A(x^n)$, then $m \geq k$ and that M^* is a subgroup of X .

We denote the class of mgroups generated from the group X by $\mathfrak{MG}(X)$

Example 2

Let $(X, \cdot)$ be a group, where $X = \mathbb{C}_2 = \{1, -1\}$. Suppose $A_0 = \{(3)1, (2)[-1]\}$ be a mset. Then A_0 is mgrou under multiplication $(\cdot)$ since:

$$(3)1 = (3)[1.1] \Rightarrow 3 \geq \min\{3, 3\} = 3$$

$$(2)[-1] = (2)[-1.1] \Rightarrow 2 \geq \min\{2, 3\} = 2$$

$$(3)1 = (3)[-1. -1] \Rightarrow 3 \geq \min\{2, 2\} = 2$$

clearly the identity $e = 1$ and $e^{-1} = 1^{-1} = 1$, $[-1]^{-1} = -1$

$(3)1 = (3)1^{-1}$ and $(2)[-1] = (2)[-1]^{-1}$. Moreover, $(3)1 \in A_0$ and $3 > 2$

where $(2)[-1] \in A_0$

Definition 8: (commutative multigroup)

Let $M \in \mathfrak{MG}(X)$. Then M_0 is said to be commutative if and only if for any $(m)[xy]$ and $(n)[yx]$ such that $m = C_M(xy)$ and $n = C_M(yx)$ then $m = n \forall x, y \in X$

Note that for any non-empty mset $M \in \mathfrak{MG}(X)$, we have M^* a subgroup of X

(Nazmul et al., 2013). However, if $M \in \mathfrak{MG}(X)$ is a commutative mgrou, then M^* does not necessarily imply a commutative subgroup of the group X . But if X is a commutative group, then M_0 is commutative.

Multiset group

Definition 9: (mset point composition) Tripathy et al. (2017)

Let X be a nonempty set. A notation $*_1$ on a nonempty mset points A_0 of $A \in \mathfrak{M}(X)$ is called an mset point composition in A_0 denoted $(m)x *_1 (n)y$ for any $(m)x, (n)y$ such that $m = C_A(x)$, $n = C_A(y)$

Definition 10: (mset point binary composition) Tripathy et al. (2017)

Let X be a nonempty set. An mset point composition $*_1$ on M_0 of $M \in \mathfrak{M}(X)$ is binary if
 $(m)x *_1 (n)y \in M_0$ for all $(m)x, (n)y \in M_0$.

Definition 11: (Associativity of mset point binary composition) Tripathy *et al.* (2017)

Let X be a nonempty set. An mset point binary composition $*_1$ on A_0 of $A \in \mathfrak{M}(X)$ is said to be associative if
 $[(m)x *_1 (n)y] *_1 (r)z == (m)x *_1 [(n)y *_1 (r)z]$
for all $(m)x, (n)y, (r)z \in A_0$

Definition 12: (Commutativity of mset point binary composition) Tripathy *et al.* (2017)

Let X be a nonempty set. A binary mset point composition $*_1$ on A_0 of $A \in \mathfrak{M}(X)$ is commutative if $(m)x *_1 (n)y == (n)y *_1 (m)x$ for all $(m)x, (n)y \in A_0$

Definition 13: (mset point identity) Tripathy *et al.* (2017)

Let X be a nonempty set and $A \in \mathfrak{M}(X)$. An mset point $(n)e \in A_0$ is called an mset point identity of A_0 under the mset point composition $*_1$ if and only if

$(n)e *_1 (m)x == (m)x == (m)x *_1 (n)e$ for all $(m)x \in A_0$.

Proposition 2: (uniqueness of mset point identity): Let X be a nonempty set and

$A \in \mathfrak{M}(X)$. The mset point identity $(n)e \in A_0$ is unique under mset point binary composition $*_1$

Proof:

Let $A \in \mathfrak{M}(X)$ and $*_1$ a binary composition on A_0

Assume $(n)e, (k)l \in A_0$ are two mset point identities of A_0 where $(n)e \neq (k)l$.

Now for any $(m)x \in A_0$ we have:

$(n)e *_1 (m)x == (m)x *_1 (n)e == (m)x$ for all $(m)x \in A_0$
(by definition 4.5) (1)

and $(k)l *_1 (m)x == (m)x *_1 (k)l == (m)x$ (by definition 4.5). (2)

Now for $(m)x == (k)l$, we have

$(n)e *_1 (k)l == (k)l *_1 (n)e == (k)l$ (from (1)) (3)

for $(m)x == (n)e$, we have

Also, $(k)l *_1 (n)e == (n)e *_1 (k)l == (n)e$ (from (2)) (4)

Thus, $(k)l == (n)e$ (from (3), (4) and proposition 2)

(5)

This contradicts the hypothesis. In particular, the identity mset point $(n)e$ must be unique.

Definition 14: (inverse mpoint): Let X be a nonempty set and $A \in \mathfrak{M}(X)$ with an mset point identity $(n)e$. An mset point $(n)y \in A_0$ is said to be the mset point inverse of the mset point $(m)x \in A_0$ if and only if $(m)x *_1 (n)y == (n)y *_1 (m)x == (n)e$ where $*_1$ is an mset point binary composition on A_0 . We denote the mset point inverse of the mset point $(m)x$ by $[(m)x]^{-1}$

Proposition 3: (uniqueness of mset point inverse): Let X be a nonempty set and

$A \in \mathfrak{M}(X)$ with an mset point identity under the associative mset

point binary composition $*_1$. The mset point inverse of an mset point of A_0 is unique.

Proof:

Let $A \in \mathfrak{M}(X)$ and $(m)x, (n)e \in A_0$ where $(n)e$ is the identity mpoint of A_0

We show that the inverse $[(m)x]^{-1} \in A_0$ is unique

Now let $(k)y, (r)z \in A_0$ be the inverse of the mpoint $(m)x \in A_0$

We show that $(k)y == (r)z$

For $(k)y$, we have $(m)x *_1 (k)y == (k)y *_1 (m)x == (n)e$ (from definition 14) (6)

For $(r)z$, we have $(m)x *_1 (r)z == (r)z *_1 (m)x == (n)e$ (from definition 14) (7)

Clearly,

$(k)y == (k)y *_1 (n)e == (k)y *_1 [(m)x *_1 (r)z] == [(k)y *_1 (m)x] *_1 (r)z$ (8)

(by definition 11)

In particular, $(k)y == (n)e *_1 (r)z == (r)z$ (from (6), (8) and definition 13) (9)

Thus, $(k)y == (r)z$ (from (9) and proposition 2)

Definition 15 (multiset group) Tripathy *et al.* (2017)

Let X be a nonempty set and $M \in \mathfrak{M}(X)$ such that $M \neq \emptyset$. Then the pair $(M_0, *_1)$ where $*_1$ is an mset point binary composition called a multiset group (mset group, for short) if the following axioms are satisfied:

- i. $[(m)x *_1 (n)y] *_1 (r)z == (m)x *_1 [(n)y *_1 (r)z]$, for all $(m)x, (n)y, (r)z \in M_0$ (associativity)
- ii. There exists $(n)e \in M_0$ such that $(n)e *_1 (m)x == (m)x == (m)x *_1 (n)e$ for all $(m)x \in M_0$ (existence of an identity mset point)
- iii. For every $(m)x \in M_0$ there exist $[(m)x]^{-1} \in M_0$ such that $[(m)x]^{-1} *_1 (m)x == (n)e == (m)x *_1 [(m)x]^{-1}$ (existence of inverse mset point)

We denote the class of all finite mset groups over the group X by $\mathfrak{M}\mathfrak{G}(X)$

Definition 16 (Binary composition defined)

Let $(X, *)$ be a nonempty group and $A \in \mathfrak{M}(X)$ such that $A \neq \emptyset$. Then a multiset point composition $*_1$ on A_0 is defined:

$(m)x *_1 (n)y := (k)[x * y]$ for any $(m)x, (n)y$ such that $m = C_A(x)$, $n = C_A(y)$ and $k = C_A(x * y)$

Clearly, $(m)x *_1 (n)y \in A_0$ if $k > 0$ and $[(m)x]^{-1} := (r)x^{-1}$ where $r = C_A(x^{-1})$

and $[(m)x]^{-1} \in A_0$ if $r > 0$.

Recall that $(m)x, (n)y \in A_0 \Rightarrow m, n > 0 \wedge x, y \in A^*$

Example 3

Let $A_0 \in \mathfrak{M}(X)$ such that $X = Z_3 = \{0, 1, 2\}$ under addition $+$ and

$A_0 = \{(4)0, (3)1, (4)2\}$ over a group X , it is easy to verify from the table below that the pair $(A_0, +_1)$ is an mset group. where $(m)x +_1 (n)y := (k)[x + y \bmod 3]$ for all $x, y \in X$

Table 1: Example of mset group

$+_1$	(4)0	(3)1	(4)2
(4)0	(4)0	(3)1	(4)2
(3)1	(3)1	(4)2	(4)0
(4)2	(4)2	(4)0	(3)1

Proposition 4

Let $(X, *)$ be a group and $M \in \mathfrak{M}(X)$ such that $M \neq \emptyset$ and $(M_0, *_1)$ a mset group. Then for every $(m)x \in M_0$ we have

- $[(m)x]^{-1}]^{-1} = (m)x$
- $[(m)x *_1 (n)y]^{-1} = [(n)y]^{-1} *_1 [(m)x]^{-1}$

Proof

Let $(M, *_1)$ be a mset group and $(m)x \in M_0$.

Thus, there exists a unique $[(m)x]^{-1} \in M_0$ such that $(m)x *_1 [(m)x]^{-1} = [(m)x]^{-1} *_1 (m)x = (n)e$ (10)

where $(n)e$ is the identity element in M (see Tella et al., 2024)

- Since $[(m)x]^{-1} \in M$, then we have $[(m)x]^{-1}]^{-1} \in M_0$ (11)
(by hypothesis and see (Tella et al., 2024))
Now $[(m)x]^{-1} := (r)x^{-1}$ where $r = C_A(x^{-1})$ (see Tella et al., 2024) (12)
Thus,
 $[(m)x]^{-1}]^{-1} = [(r)x^{-1}]^{-1} = (k)[x^{-1}]^{-1} = (k)x \in M_0$ (from (12)) (13)
But $(m)x, (k)x \in M_0 \Rightarrow k = m$ (uniqueness of multiplicity) (14)
In particular $(m)x = (k)x$ and $[(m)x]^{-1}]^{-1} = (m)x$ (from (13) and (14))
ii. $[(m)x *_1 (n)y] *_1 [(n)y]^{-1} *_1 [(m)x]^{-1}$
 $= (m)x *_1 [(n)y] *_1 [(n)y]^{-1} *_1 [(m)x]^{-1}$ (15)
(by hypothesis and see (Tella et al., 2024))
But $(m)x *_1 [(n)y] *_1 [(n)y]^{-1} *_1 [(m)x]^{-1}$
 $= (m)x *_1 (n)e *_1 [(m)x]^{-1}$ (by hypothesis and see (Tella et al., 2024)) (16)
 $= (m)x *_1 [(m)x]^{-1} = (n)e$ (by hypothesis and see (Tella et al., 2024)) (17)
Similarly, $[(n)y]^{-1} *_1 [(m)x]^{-1} *_1 [(m)x *_1 (n)y]$ (18)
(by hypothesis and see (Tella et al., 2024))
 $= [(n)y]^{-1} *_1 [(m)x]^{-1} *_1 (m)x *_1 (n)y$ (19)
(by hypothesis and see (Tella et al., 2024))
 $= [(n)y]^{-1} *_1 (n)e *_1 (n)y$ (by hypothesis and see (Tella et al., 2024)) (20)
 $= [(n)y]^{-1} *_1 (n)y = (n)e$
Thus, $[(n)y]^{-1} *_1 [(m)x]^{-1} = [(m)x *_1 (n)y]^{-1}$.
i.e, $[(m)x *_1 (n)y]^{-1} = [(n)y]^{-1} *_1 [(m)x]^{-1}$

Definition 17 (mset subgroup) Tripathy et al. (2017)

Let $M \in \mathfrak{M}(X)$ such that $(M_0, *_1)$ is an mset group and $N \subseteq M$. Then $(N_0, *_1)$ is an mset subgroup of $(M_0, *_1)$ if and only if $(N_0, *_1)$ is an mset group.

Example 4

Let $A \in \mathfrak{M}(X)$ such that $X = Z_5 = \{0, 1, 2, 3, 4\}$ under addition $+$ and (i)
given an mset group $A_0 = \{(4)0, (3)1, (3)2, (3)3, (2)4\}$

under the mset point binary composition $+_1$ and $B_0 = \{(2)0, (2)1, (2)2, (1)3, (2)4\}$

Clearly $B_0 \subseteq A_0$, and $(B_0, +_1)$ is an mset subgroup

Theorem 1 Tripathy et al. (2017)

Let $M \in \mathfrak{M}(X)$ such that $(M_0, *_1)$ is an mset group and $N \subseteq M$. Then $(N_0, *_1)$ is a mset subgroup if and only if

- $*_1$ is a binary composition on N_0 .
- The identity mset point $(n_e)e \in M_0$ is such that $(k_e)e \in N_0$ and $k_e \leq n_e$
- If $(m)x \in N_0$, then $[(m)x]^{-1} \in N_0$

Theorem 2 Tripathy et al. (2017)

Let $(X, *)$ be a group and $M \in \mathfrak{M}(X)$ such that $(M_0, *_1)$ is an mset group. Then $(N_0, *_1)$ is a mset subgroup if and only if for any $(m)x, (n)y \in N_0$ we have $(m)x \odot [(n)y]^{-1} \in N_0$

Proposition 5

Let $(X, *)$ be a nonempty group and $A \in \mathfrak{M}(X)$. Then $(A_0, *_1)$ is commutative if and only if $(A^*, *)$ is commutative.

Proof

Assume that $(A_0, *_1)$ is commutative. Then for any $(m)x, (n)y \in A_0$

$(m)x *_1 (n)y = (n)y *_1 (m)x$ (by definition 2.4) (21)

But $(m)x *_1 (n)y = k[x * y]$ where $k \geq 0$ (see Tella et al., 2024) (22)

and $(n)y *_1 (m)x = (r)[y * x]$ where $r \geq 0$ (see Tella et al., 2024) (23)

But $(m)x *_1 (n)y = (n)y \odot (m)x \Rightarrow (k)[x * y] = (r)[y * x]$ (24)

(from (22) and (23))

In particular, $k = r$ and $x * y = y * x$ (from (24)) (25)

But $(m)x, (n)y \in A \Rightarrow x, y \in A^*$ (by definitions 1 and 3) (26)

Thus, $x, y \in A^* \Rightarrow x * y = y * x$ (from (25) and (26))

In particular, $(A^*, *)$ is commutative

Conversely, Let $(A^*, *)$ be commutative and $x, y \in A^*$. Then there exists $m, n > 0$

such that $(m)x, (n)y \in A_0$ (by definition 2.1 and 2.4) (27)

But $(m)x *_1 (n)y = (k)[x * y]$ where $k \geq 0$ (from (22))

and $x * y = y * x$ (by hypothesis) (28)

Thus, $(k)[x * y] = (k)[y * x]$ (From (28) and by Blizard (1989)) (29)

In particular, $(m)x *_1 (n)y = (n)y *_1 (m)x$ (from (22), (23) and (29))

Thus, $(A_0, *_1)$ is commutative

RESULTS

Multigroup as Multiset group

Proposition 6: Let $(X, *)$ be a nonempty group. Then $A \in \mathfrak{M}(X) \Rightarrow A \in \mathfrak{M}(X)$

Proof

Let $A \in \mathfrak{M}(X)$ with the mpoint binary composition $*_1$ and $(m)x, (n)y \in A_0$

Then we show that

for any $(m)x, (n)y \in A_0$, we have $(m)x *_1 (n)y \in A_0$

Now $(m)x, (n)y \in A_0 \Rightarrow m, n > 0$ (from definition 1)

(30)
In particular, $(m)x, (n)y \in A_0 \Rightarrow \min\{m, n\} > 0$ (from (30))
(31)
Thus, $(m)x *_1 (n)y := k[x * y] \Rightarrow k \geq \min\{m, n\} > 0$
(32)
(see (Tella et al., 2024) and from (31))
In particular, $k[x * y] \in A_0$ and $(m)x *_1 (n)y \in A_0$
(33)
(by definition 1 and from (32))
Thus, $*_1$ is binary.
Let $(m)x, (n)y, (k)z \in A_0$
Clearly, $(m)x *_1 [(n)y *_1 (k)z] == (m)x *_1 (r)[y * z]$
(34)
where $r \geq \min\{n, k\} > 0$ (from definition 1)
But $(m)x *_1 (r)[y * z] := (t)[x * [y * z]]$
(35)
((from definitions 1, 9 and 11)) where $t \geq \min\{m, r\} = \min\{m, \min\{n, k\}\} > 0$
Thus, $(m)x *_1 [(n)y *_1 (k)z] = (t)[x * [y * z]] \in A_0$
(36)
(from (34) and (35))
But $[x * [y * z]] = [[x * y] * z]$ (by hypothesis on X)
(37)
Thus $(t)[x * [y * z]] == (t)[[x * y] * z]$
(38)
(unique multiplicity from (36) and (37))
In particular, $(m)x *_1 [(n)y *_1 (k)z] == [(m)x *_1 (n)y] *_1 (k)z$ (39)
(from (34) and (38)),
(ii) let $(m)x \in A_0$ and e be the identity in X such that $n_e = C_A(e)$ and $m = C_A(x)$
Clearly $m > 0$ and $n_e \geq m > 0$ (by hypothesis and see (Tella et al., 2024)) (40)
In particular, $(n_e)e \in A_0$ (by definition 1)
(41)
Now for any $(m)x \in A_0$, we have $(m)x *_1 (n_e)e := (k)[x * e]$ (42)
where $k \geq \min\{m, n_e\} = m$ (see (Tella et al., 2024) and from (41))
But $x * e = x$ (existence of identity in X by hypothesis)
(43)
Thus, $(m)x *_1 (n_e)e := (k)x \in A_0$ (from (42) and (43))
(44)
 $(m)x, (k)x \in A_0 \Rightarrow m = k$ (from Blizard (1989))
(45)
(by hypothesis) and from (44))
In particular, $(m)x *_1 (n_e)e == (m)x$ (from (44) and (45))
(46)
Thus, the existence of an mset point identity $(n_e)e \in A_0$ (from (46)) (47)
(iii)
let $(m)x \in A$ where $m = C_A(x)$.
Then we have $x \in X$ and $x^{-1} \in X$ such that $x * x^{-1} = e$ (by hypothesis on X) (48)
But for $n = C_A(x^{-1})$, we have $n = m > 0$
(49)
(by hypothesis and from definition 1)
Thus, $(m)x^{-1} \in A_0$ and $(m)x *_1 (m)x^{-1} := (k)[x * x^{-1}] == (k)e$ (50)

(from (49), (48) and by definition 1)
In particular, $(m)x *_1 (m)x^{-1} == (k)e \in A_0$
(51)
where $k = \min\{m, m\} = m > 0$ (see (Tella et al., 2024) and from (49))
But $(n_e)e, (k)e \in A_0 \Rightarrow n_e = k$ (unique multiplicity in Blizard(1989)) (52)
Hence, $(m)x *_1 (n)x^{-1} == (n_e)e$ (from (51) and (52))
(53)
Existence of an inverse mset point (from (52))
Thus, $A \in \mathfrak{MS}(X)$ (from (i), (ii) and (iii) above)
However, the converse of Proposition 6 does not hold, using Example 3
(since $3 \geq \min\{4, 4\} = 4$ where $(4)2 +_1 (4)2 = (3)1$)

Conclusion

This paper presents a conceptual unification of the structures of multigroups and multiset groups. The unified approach, termed "mgrouop set," reduces the count-function emphasis of an mgrouop to the term-emphasis of an mset group. By this approach, the necessary and sufficient condition for a commutative mgrouop set is hereby established via the commutativity of its root set. However, this new paradigm highlights the theoretical relevance of groups in general and opens the door to analogous study in multiset theory.

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