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COMPARATIVE ANALYSIS OF THE AFFINE SCALING AND KARMARKAR'S POLYNOMIAL – TIME FOR LINEAR PROGRAMMING

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ABSTRACT

The simplex method is the well-known, non-polynomial solution technique for linear programming problems. However, some computational testing has shown that the Karmarkar's polynomial projective interior point method may perform better than the simplex method on many classes of problems, especially, on problems with large sizes. The affine scaling algorithm is a variant of the Karmarkar's algorithms. In this paper, we compare the affine scaling and the Karmarkar algorithms using the same test LP problem.

***Keywords*:** Polynomial-time, Complexity bound, Primal LP, Dual LP, Basic Solution, Degenerate Solution, Affine Space, Simplex and Polytope.

INTRODUCTION

In 1979 Khachian proposed the ellipsoid method to solve LP problems in polynomial-time. He adapted the ellipsoid method used in convex optimization developed independently by Shor (1968) and Ludin & Nemirovskii (1976) to give a polynomial-time algorithm for LP. The complexity bound of his algorithm was $0(n^4L)$, where n is the number of variables and L is the input length. The practical implementation of the Khachian (1979) algorithm has been inefficient.

In 1984, a polynomial-time linear programming algorithm using an interior point method was proposed by Karmarkar (1984). However, interior point methods were known as early as the 1960s in the form of the barrier function methods, but the media publicity that accompanied Karmarkar's announcement led to these methods receiving a great deal of attention. The computational complexity of Karmarkar (1984) algorithm in theoretical terms was better than Khachian's algorithm by a factor of $0(n^{0.5}L)$, since the complexity bound of Karmarkar (1984) was $0(n^{3.3}L)$. Karmarkar's algorithm compares favourably with the simplex method. In fact, it has been observed in Karmarkar (1984), that the United State Air Force independently tested Bell Lab Production Code which implements Karmarkar's projective interior point method against the simplex method, using the same machine and the same test problems and found that Karmarkar's method was 400 times faster than the simplex method on the largest problem they could solve by the simplex method, but more significantly, they were able to solve even larger problems, which previously were unsolvable due to the limitations of the simplex method.

Current efficient implementations of the interior point methods are mostly based on a predictor-corrector technique (Mehrotra & Sun 1987), where the Cholesky decomposition of the normal equation or LDL^T factorization of the symmetric indefinite augmented system is used to perform Newton's method (together with some heuristics to estimate the penalty parameter).

In this paper, we described the affine scaling algorithm and the Karmarkar's algorithm and ended up by comparing the two.

The Affine Rescaling Algorithm

This method was first introduced by Dikin (1967) and later re-introduced by Barnes (1996) and Vanderbei *et al*. (1986).

The Algorithm:

It starts directly on the LP problem in standard form:

$$\begin{aligned} &\text{minimize } z = c^T x \\ &\text{subject to } Ax = 6 \\ &\qquad x \geq 0 \end{aligned} \qquad \ldots(1.0)$$

The algorithm generates a monotonic decreasing sequence of the objective function value.

We derive the algorithm based on Barnes (1996) approach. The dual of (1.0) is

$$\begin{aligned} &\text{maximize } z\ b^T w \\ &\text{subject to } A^T w \geq c \\ &\qquad w \geq 0 \end{aligned} \qquad \ldots (2.0)$$

Barnes (1996) made the following assumptions:

i. That the prima LP (1.0) has no degenerate basic feasible solution.
ii. That no basic solution exists for the dual LP (2.0)

Hence, from the assumption (i), b cannot be expressed as a positive combination of fewer than m columns of A and from the assumption (ii), at most m of the equations

$c_i - a_i^T \lambda = 0,\ i = 1, 2, ..., m$ can be satisfied simultaneously.

The LP problems (1.0) and (2.0) remain non-degenerate under small perturbations in b and c. In fact, if LP (2.0) is non-degenerate, then there exist a number $\in > 0$ such that any feasible solution of (1.0) will have at least m components greater than $\in_1$. Also, if the dual LP (2.0) is non-degenerate, then there exist a number $\notin_2 > 0,$ such that at most m of the inequalities $\left|c_i - a_i^T \lambda\right| < \in\!\!\!/_2$ can be satisfied simultaneously.

$\in_1$ and $\in_2$ can be found such that the above conditions hold when b and c are perturbed slightly.

Now, let $(y_1, \ldots, y_n)^T$ be a feasible solution of the primal LP (1.0) that satisfies $y_1 > 0,\ j = 1, 2, \ldots, n.$ If $0 < R < 1,$ then the ellipsoid

$$\sum_{j=1}^{n} \frac{(x_j - y_1)^2}{y_j^2} \le R^2 \qquad \ldots(3.0)$$

lies in the interior of the positive orthant P_+ in E_0 (Barnes 1996).

Hence, a feasible solution of LP (1.0) that satisfies $c^T x < c^T y$ can be obtained by solving the following LP problem:

$$\text{minimize } z = c^T x$$
$$\text{subject to } Ax = b$$
$$\sum_{j=1}^{n} \left(\frac{x_j - y_j}{y_j}\right)^2 \le R^2 \qquad \ldots(4.0)$$

Hence, the constraint $x \ge 0$ in (1.0) can be replaced by (3.0), which is easier to handle (Barnes 1996).

Now, to solve (4.0), we let $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ be a vector of lagrange multipliers that correspond to the constraint $Ax = b.$
Let $D = diag\ (y_1, \ldots, y_n)$. Since $y_j > 0,\ D$ is non-singular.
Now, for any x that satisfies (4.0), we have

$$\begin{aligned} c^T y - c^T x &= \left\{c - A^T \lambda\right\}^T (x - y) \\ &= \left|D\left(c - a^T \lambda\right)^T\ D^{-1}\ (x - y)\right| \\ &\le \left|D\left(c - A^T \lambda\right)^T\right| \left|D^{-1}\ (x - y)\right| \\ &= \left|D\left(c - a^T \lambda\right)\right| R \end{aligned} \qquad \ldots (5.0)$$

The inequalities in (5.0) are obtained from (2.0) and Schwartz's inequality. Equality holds in (5.0) if

$$\left(c - A^T \lambda\right) = \gamma\left(D^{-1}(x - y)\right) \qquad \ldots (6.0)$$

for some constants γ and if $\left|D^{-1}(x - y)\right| = \gamma^2,$ which implies that

$$\gamma = \left|D\ (c - A^T \lambda)\right| \qquad \ldots (7.0)$$

substituting (7.0) in (5.0) implies that

$$D\left(c - A^T \lambda\right) = \frac{\left|D\left(c - A^T \lambda\right) D^{-1}\right| (x - y)}{R}$$

Thus, $x,\ y - \dfrac{RD^2\left(c - A^T \lambda\right)}{\left\|D\left(c - A^T \lambda\right)\right\|}$ ... (8.0)

The condition $Ax = Ay = b$ implies that

$$\lambda = \left(AD^2 c\right)\left(AD^2 A^T\right)^{-1} \qquad \ldots (9.0)$$

Now from (5.0), it implies that

$$c^T x \ge c^T y - R\left|D\left(c - A^T \lambda\right)\right| \qquad \ldots (10.0)$$

Hence, the minimum of (10.0) is given by its right hand side and it is obtained when x and λ are given by (8.0) and (9.0) respectively. This suggests an algorithm for iteratively finding the solution of the primal problem (1.0).

The Algorithm according to Barnes (1996) states as follows:
Let $x^0 > 0$ that satisfies $Ax^0 = b$ can be given. In general, if x^k is known, define

$$D_k = diag\ (x_1^k, x_2^k, \ldots, x_n^k)$$

and compute $x^{k+1} > 0$ by the following formulae:

$$x^{k+1} = x^k - \frac{RD_k^2\left(c - A^T \lambda_k\right)}{\left\|D_k\left(c - A^T \lambda_k\right)\right\|}$$
$$\text{where } \lambda_k = \left(AD_k^2 C\right)\left(AD_k^2 A^T\right)^{-1} \qquad \ldots(11.0)$$

Theorem 1
If the primal LP (1.0) has a bounded solution, then the sequence $\left\{x^k\right\}$ defined by (11.0) converges to a solution of (1.0), that is an extreme point of the constraint set defined by $Ax = b,\ x \ge 0$
Proof (see Barnes 1996)

Theorem 2
Let x^* denote the solution of the primal LP (1.0). Sequence $\left\{x^k\right\}$ generated by the dual LP (2.0) satisfies

$$c^T x^{k+1} - c^T x^* \le \left(1 - \frac{R}{\sqrt{n - m + E_k}}\right)\left(c^T x^k - c^T x^\bullet\right) \qquad \ldots (12.0)$$

where $\left\{\in_k\right\}$ is a sequence of positive numbers converging to 0 as $k \to \infty$. Proof (see Barnes 1996)

From the non-degenerating assumptions, we know that

$$x^* = \lim_{k \to \infty} x^k$$

has (n – m) components equal to zero.

For simplicity, we assume that

$$x^* = \left(x_1^*, x_2^*, \ldots, x_n^\bullet, \ldots, 0\right)$$

Hence, from (6.0),

$$c^T x^k - c^T x^\bullet \le \left| D_k\left(c - A^T\lambda_k\right)^T \right| \left| D_k^{-1}\left(x^k - x^\bullet\right) \right|$$

$$\le \left\| D_k\left(c - A^T\lambda_k\right) \right\| \left(n - m + \in_k\right)^{1/2}$$

$$= \frac{1}{R}\left(c^T x^k - c^T x^{k+1}\right)\left(n - m + \in_k\right)^{1/2}$$

where $\in_k = \sum_{k=0}^{n}\left(\frac{x_i^k - x_i^\bullet}{x_i^k}\right)$ ...(13.0)

which tends to zero as $k \to \infty$. Hence, (2.13) can be written as

$$\frac{R\left(c^T x^k - c^T x^\bullet\right)}{\sqrt{n - m + \in_k}} \le c^T x^k - c^T x^{k+1} - c^T x^\bullet$$

i.e $c^T x^{k+1} - c^T x^\bullet - \left(c^T x^k - c^T x^\bullet\right) \le \frac{-R\left(c^T x^k - c^T x^\bullet\right)}{\sqrt{n - m + \in_k}}$

which is equivalent to (12.0). From the above theorem 2, Barnes 1996) noted that the amount by which the objective function $c^T x$ decreases at each iteration of the algorithm increases, if R is increased. This suggests that, at each step of the algorithm, R should be increased as much as possible, with the condition that all the variables remain non-degenerate.

The affine scaling algorithm has been successfully implemented by Adler *et al.* (1991) and Monma & Morton (1987).

Karmarkar's Algorithm.

Let $\Omega = \{x | Ax = 0\}$

$\Delta = \{x | \sum x_i = 1,\ x \ge 0\}$

and $\Pi = \Omega \cap \Delta$

be the affine space, simplex and polytope respectively.

Karmarkar (1984) considered an LP problem in standard forms

minimize $z = c^{YT} x$

subject to $Ax = b$

$x \ge 0$

where $c,\ x \in z^n,\ a \in z^{mxn}$ and $b \in z^n$. ... (14.0)

On application of some projective transformation on (1.0), Karmarkar (1984) obtained the following canonical form LP:

maximize $z = c^T x$

subject to $Ax = 0$

$e^T x = 1$

$x \ge 0$

where $e^T = (1, 1, .., 1)$

The target minimum value of the objective function in this algorithm is zero.

The algorithm:

It operates in such a way that it generates a sequence of points $x^0, x^{(1)}, \ldots x^k, ..$ in the following steps:

Step 1: **Initialization:**

set $x^{(0)}$ to the centre of the simplex $\Delta = \left(\frac{1}{n}e\right)$

Step 2: **Computation of the next point:**

Let $b = x^{(k+1)}$ and $a = x^k$ is defined by the following sequence of operation:

Let $D = diag\,(a_i, \ldots, a_n)$, where the i^{th} entry is a_i. Let $B = \begin{bmatrix} AD \\ e^T \end{bmatrix}$, i.e. augment the matrix AD with a row of all 1's. This is to guarantee that $\ker B$ (i.e: the null space of B) is contained in the hyperplane $\{x | \sum x_i = 1,\ x \ge 0\}$.

Compute the orthogonal projection of DC into the null space of B *i.e* $c_p = \left\{1 - B^T\left(BB^T\right)^{-1}B\right\}Dc$

Normalize $\hat{c} = \frac{c_p}{\|c_p\|}$ i.e; $\hat{c}$ is the unit vector in the direction of c_p.

Let $B = a_0 - \alpha r\hat{c}$, i.e; take a step of length αr in the direction of $\hat{c}$, where $r = \frac{1}{\sqrt{n(n-1)}}$ is the radius of the largest sphere and $\alpha \in (0, 1)$ (a parameter which Karmarkar set equal to $1/4$).

Apply the inverse projective transformation to b i.e. $b = \frac{Db^1}{e^T Db^1}$.

Return.

Step 3: **Check for feasibility.**

Karmarkar (1984) defined a potential function $f(x) = \sum \ln \frac{c^T x}{x_i}$ in which a certain improvement δ in the potential function is expected at each iteration. The value of δ depends on the choice of the parameter α, e.g. if $\alpha = 1/4$, then $\delta = 1/8$ (Karmarkar 1984). If the expected improvement is not obtained i.e. if $f\left(x^{(k+1)}\right) > f\left(x^{k(k)}\right) - \delta$, then, we stop and conclude that the minimum value of the objective function of the LP problem (2.1), problem from which we obtained the canonical form does not have a finite optimal solution i.e, it is either infeasible or unbounded.

Step 4: **The termination Rule:**

The algorithm stops, when the required convergence check is obtained.

i.e. when $\frac{c^T x}{c^T a_0} \le 2^{-q}$

where q is a positive real number. Standard form variants of Karmarkar's algorithm have been developed by Asntreicher (1985); Gay (1987); Steiger (1988); Ye & Kojima (1987).

Comparison

Karmarkar's algorithm is a polynomially bounded interior point method. To the best of our knowledge, no general results on the bounds on the efficiency of the affine scaling algorithm is available, but due to analysis carried out by Megiddo & Shub (1986), about associated continuous trajectory, it is suspected that primal-dual affine scaling algorithms are exponential in the worst cases. However, Monteiro & Adler (1990) have shown that one primal-dual version of the affine scaling algorithm has a polynomial-time complexity, provided that it starts near the 'center' of the feasible set and takes sufficiently small step-size. It is still an open question if possible to achieve polynomial bound for this variant of the affine scaling algorithm under less restrictive conditions or not.

The affine scaling algorithm has the following advantages over the original Karmarkar's algorithm. It starts on the LP problem in standard form and assumes that a point x^0 is known such that $Ax^0 = b,\ x^0 \geq 0$. It generates a monotonic decreasing sequence of the objective function values and the minimum of the objective function need not be known in advance. The affine scaling algorithm is one of the simplest and most efficient polynomial time algorithms for solving LP problems. It is very attractive due to its simplicity and its excellent performance in practice.

For both the affine scaling and Karmarkar's algorithms, like any other interior point algorithms, the bulk of the computation work are concentrated in the projective operation needed in each of the iterations.

It is generally believed that the proof of convergence of affine-scaling algorithms in the absence of degeneracy is fairly straight forward, while under degeneracy, such a proof is long and cumbersome. Manual computations of both algorithms are repetitive, cumbersome and prone to errors.

Experimental results have shown that the affine scaling algorithms compete favourably with the MINOS code which implements the simplex algorithm for linear programming.

Example:

We solve the following problem manually by the use of both methods:

minimize $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

For the affine scaling algorithm, we let $x^0 = (2,\ 2,\ 2,\ 9)$ be an interior point that satisfies $Ax^0 = b,\ x^0 \geq 0$ and for the Karmarkar's algorithm, we choose $a_1 = 2, a_2 = 1, a_3\ 2, a_4 = 3, a_5 = 1$ and $a_b = 1$ as the strictly interior points in the positive orthant, that satisfy the LP upon introduction of artificial variables and a_i, subsequently replacing x_i.

Conclusion

The simplex method gave the most accurate result of $z = 2.40$ to the LP under consideration. We note that the other methods especially the Karmarkar's algorithm gave solution that were a bit far from that of the simplex method. At the 8^{th} iteration, the affine scaling algorithm gave its solution as $z = 2.42$, while Karmarkar's result even at the 8^{th} iteration when optimality was reached, (i.e. when $c^T x = 0$) gave its result as $z = 2.50$.

There were numerous round-off errors involved when applying the algorithms manually, especially for the Karmarkar's algorithm.

We note that it is needless using the interior point algorithms to solve small-size LP problems, since they are even more tedious, time consuming and boring as the number of iterations increases. We therefore recommend their use only for large LP problems and using computer softwares.

Furthermore, the affine scaling method was found to be relatively simpler than the Karmarkar's algorithm and gave the closer result to that of the simplex method.

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