FULL LENGTH RESEARCH ARTICLE

LOCALISATION IN COMMUTATIVE RINGS AT A PRIME IDEAL $p = (0), (2)$

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ABSTRACT

This paper discuses localization, one of the most important concepts in commutative algebra. A process by which new rings are constructed. The nature of the resulting new ring depends on that of a multiplicative subset of the given ring. This paper also focuses on the extension of the ring of integers Z to that of rational numbers Q that is embedding the integers into the rational numbers. Some results are stated in this paper.

Keywords: localization, rings

INTRODUCTION

The formation of rings of fractions (new rings) and the associated process of localization are the most important technical tools in commutative algebra. The geometric notion of concentrating attention near a point has as the algebraic analogue the important processes of localizing a ring at the prime ideal p. Commutative algebra is essentially the study of commutative rings. It developed from two sources:

i algebraic geometry which began with the study of polynomial rings in finitely many indeterminates over a field k;

ii algebraic number theory which began with the study of the ring of rational numbers, the central notion in commutative algebra is that of a prime ideal which provides a common generalization of the primes of arithmetic and the points of geometry. The most important properties of localization are that: it preserves exactness and the Noetherian property. In his paper R is a commutative ring with identity and p, a prime ideal.

Definitions of fundamental concepts:

Definition 1 (Jacob 1969)
A non-empty subset I of a commutative ring R with unity is called an ideal if it is an additive subgroup of R and it is such that $RI \subseteq I$, that is for any $x \in R, y \in I$ then $xy \in I$.

Definition 2 (Kuku 1980)
An ideal $I \in R$ is called a prime ideal if

$i.$ $I \neq (1)$,
$ii.$ $xy \in I \to x \in I$ or $y \in I$. In this paper we denote this ideal by p.

Definition 3
An ideal $I \neq (1) \in R$ is called a maximal ideal if there does not exist any other ideal $J \neq (1)$ such that $I \subseteq J \subseteq R$.

Definition 4 (Jacob 1969)
Localization is a technique or method in modern ring theory by which new rings are constructed by concentrating attention near a point (or prime) p.

Definition 5 (Jacob 1969)
A subset T of a ring R is called a multiplicative subset of R if: $1 \in T$ and $a, b \in T$ implies $ab \in T$.

Definition 6 (Cohn 1977)
Let R be a ring and T a subset of R, a homomorphism $\alpha : R \to R'$ is $T - inverting$ if α maps the elements of T to invertible elements of R'.

Definition 7
The ring R is called a local ring if it has exactly one maximal ideal.

Definition 8
An element $x \in R$ is said to be a unit or an invertible in R if x divides 1, that is there exists an element $y \in R$ such that $xy = 1$.

Definition 9
A ring D is said to be embedded in a ring R if there is an isomorphism β of D onto a sub ring R' of R then R is called an extension or an overing of D. β is called an embedding of D onto R.

Definition 10 (Cohn 1977)
Suppose R is an integral domain and T a multiplicative subset of R. We construct a ring of quotient or field of fraction from R as an extension of the process involve in construction of the rational field Q from the integers Z with respect to T. The relation on RXT is defined by considering the set $\{\frac{r}{t} : r \in R, t \in T\}$ and let $\frac{r}{t}$ and $\frac{r_1}{t_1}$ be equivalent, then there exists $u \in T$ such

$(r,t) \sim (r_1,t_1) \Leftrightarrow (rt_1 - tr_1)u = 0 ------(1)$ for some $u \in T$ .This is clearly an equivalence relation defined on RXT .

We claim that T consists of entirely non-zero divisors, then (1) becomes $rt_1 - tr_1 = 0 -----(2)$ or just $rt_1 = tr_1$ and (1) defines an equivalence relation on RXT .

Now let $\frac{r}{t}$ be the equivalence class containing (r,t) and $T^{-1}R$ or R_t, the set of distinct equivalence classes of (r,t), that is an element of R_t has the form $\frac{r}{t}, r \in R, t \in T$. We make R_t into a ring as follows: $\frac{r}{t} + \frac{r_1}{t_1} = \frac{rt_1 + tr_1}{tt_1}$ and $\frac{r}{t} . \frac{r_1}{t_1} = \frac{rr_1}{tt_1}$, $t, t_1 \neq 0$. As the operations $(+)$ and $(.)$ are well defined we have that R_t is a ring with $\frac{0}{1}$ and $\frac{1}{1}$ as the zero and unit elements respectively. This new ring R_t is called the ring of fraction of R by T or ring of fraction of R with respect to T or simply the ring of fractions (with denominators in T).

The elements $\frac{r}{t}$ of R_t are constructed from the elements a say of R hence there exists a natural homomorphism or mapping β from R to R_t denoted by $\beta : R \to R_t$ defined by $\beta(r) = \frac{r}{1}$ or $\beta : r \to \frac{r}{1}$ .This implies that β maps each elements of T to a unit in R_t, that is the homomorphism is $T-inverting$.

Remarks

(i) The ring R_t under the operations defined above is a commutative ring with identity called the ring of fractions.

(ii) The homomorphism $\beta : R \to R_t$ defined by $\beta : r \to \frac{r}{t}$ is not injective in general.

If R is an integral domain and T the set of all non-zero elements (non-zero divisors) of R, that is $T = R - \{0\}$ then the homomorphism $\beta : R \to R_t$ is injective hence R_t is in particular the field of fractions, $\{0\} \neq R$.

(iii) If T does not contain the zero element that is $0 \notin T$ then (1) becomes $(r,t) \sim (0,1) \Rightarrow (r.1 - t.0)0 = 0$ hence: $\frac{r}{t} = 0$, r and t as defined in (1) and so $R_t = 0$. This result is trivial.

(iv). Where R consists of all non-zero divisors, then R_t is called the total ring of fractions. However, this is not so if R is non-commutative.

Example

For every $d \in R$ the set of $d^n, n \in N$ is a multiplicative subset of R .

Proposition (Atiyah & Macdonald 1969)

(1). If $0 \in T$ then $R_t = 0$

(2) θ is injective if and only if T contains no zero divisors given $\theta : R \to R_t$.

Proof: (1). Recall that the zero element of R_t is $\frac{0}{1}$ as $\frac{r}{t} + \frac{0}{1} = \frac{r}{t}$ .If $0 \in T$, and $\frac{r}{t} \in R$, then $\frac{r}{t} = \frac{0}{1}$ since $0(r1 - 0t) = 0$. Hence R_t reduces to just the zero- element.

(3) $r \in \ker\theta$ if and only if $\theta(r) = \frac{r}{1} = 0 \in R_t$ if and only if there exists $t \in T$ such that $tr = 0$. Thus $\ker\phi = 0$ if and only if T contains no zero element.

Definition 11 (Cohn 1977)

Let R be a commutative ring and T a multiplicative subset of R . A homomorphism $\lambda : R \to R_t$ is said to be universal $T-inverting$ if it is $T-inverting$ and for every $T-inverting$ homomorphism $\beta : R \to R'$ there exist a unique homomorphism $\beta' : R_t \to R'$ such that $\beta = \lambda\beta'$. This property determines R_t up to isomorphism. The elements of R_t can be written as fractions $\frac{r}{t}, (r \in R, t \in T)$ where $\frac{r}{t} = \frac{r_1}{t_1}$ if and only if $(rt_1 - tr_1)u = 0$ for some $u \in T$ and $\ker\lambda = \{r \to R : ru = 0, u \in S\}$

Proposition (Cohn 1977)

Let R be a commutative ring and T a multiplicative subset of R. There exist a ring R_t and a homomorphism $\lambda : R \to R_t$ which is universal $T-inverting$.

Proof: We observe that λ is $T-inverting$. Next let $\beta : R \to R'$ be any $T-inverting$ homomorphism and defined a mapping $\beta' : RXT \to R'$ by $(r,t)\beta' = (r\beta)(t\beta)^{-1}$. This holds as β is $T-inverting$, β' takes the same values in equivalent pairs: if $(rt_1 - tr_1)u = 0$, then

$(r\beta.t_1\beta - t\beta.r_1\beta)u\beta = 0$ and hence $r\beta(t\beta)^{-1} = r_1\beta.(t_1\beta)^{-1}$. We obtained a well defined mapping $\beta^{'} : R_t \to R^{'}$. This mapping has the property that $(\frac{r}{1})\beta^{'} = r\beta$ *. That is $\lambda\beta^{'} = \beta$ and it is the only such mapping for the equation * determines values of $\beta^{'}$ on the elements of $\frac{r}{1}$ and its value on $\frac{1}{t}$ must then be the inverse of its value on $\frac{t}{1}$. R_t is unique by universality. Finally, we take $r \in \ker\lambda$, this implies that: $\frac{r}{1} = \frac{0}{1}$, that is for some $ru = 0$ for $u \in T$.

Remark

Where T does not contain the zero element, then $R_t \neq 0$ and (2) fails to hold.

If $0 \notin T$ where T is the compliment of a prime ideal p, we replace t in R_t with p to obtain R_p.Note that as $0 \notin T$, then $0 \in p$. So the prime ideal p of R here corresponds to the unique maximal ideal of R_p containing all non-units.

The ring R_p just constructed is a local ring called the local ring of R at p. The process of constructing R_p is called localization and so R_p is the localization of R at the prime ideal p.

Definition 12

Let R be a commutative ring with identity, p a prime ideal of R and $R - p$ is a multiplicative subset of R. Then the ring of fractions R_p is a local ring.

Remark

No confusion should arise in the use of R/p and R_p as in the former ordinarily speaking is obtained by 'putting the elements in p equal to zero while the latter is obtained by making the elements outside p invertible.

Properties of the ring R_p and the homomorphism $\beta : R \to R_p$:

1. $t \in T \Rightarrow \beta(t)$ is a unit in R_p.
2. $\beta(r) = 0 \Rightarrow rt = 0$ for some $t \in T$.
3. Every element of R_p is of the form $\beta(r)\beta(t)^{-1}$ for some $r \in R, t \in T$.

These conditions determine R_p up to isomorphism and we state precisely the following proposition without proof.

Proposition

If $\alpha : R \to R^{'}$ is a ring homomorphism such that:

i. $t \in T \Rightarrow \alpha(t)$ is a unit in $R^{'}$

ii. $\alpha(r) = 0 \Rightarrow rt = 0$ for some $t \in T$.

iii. Every element of $R^{'}$ is of the form $\alpha(r)\alpha(t)^{-1}$

Then there is a unique homomorphism $\pi : R_p \to R'$ such that $\alpha = \pi.\beta$.

Proposition (Atiyah & Macdonald 1969)

Let p be a prime ideal of R then $T = R - p$ is multiplicatively closed and R_p is a local ring.

Proof: Elements $\frac{r}{t}$ with $r \in p$ form an ideal m in R_p. If $\frac{b}{c} \notin m$ then $b \notin p$ hence $b \in T$.

And so $\frac{b}{c}$ is a unit in R_p. Hence, if q is an ideal in R_p and $q \not\subset m$, then q contains a unit and hence is the whole ring. Therefore m is the only maximal ideal in R_p that is, R_p is a local ring.

Proposition

To every ideal I in R we associate the expanded ideal I_t of R_t generated by the image $\lambda : I_t = \{\frac{i}{t} : i \in I, t \in T\}$. We need to show that I_t is an ideal.

Proof: If $\frac{i}{t}, \frac{i_1}{t_1} \in I_t$, then $\frac{i}{t} + \frac{i_1}{t_1} = \frac{(it_1 + ti_1)}{tt_1} \in I_t$ and for any $\frac{h}{k} \in R_t, \frac{i}{t}.\frac{h}{k} = \frac{ih}{tk} \in I_t$ hence I_t is an ideal. This theorem shows the relationship between ideals in R and R_t.

Conclusion

In conclusion we state the following results, ending with the significance of localization.

1. If $p = (2)$ and $R = Z$, the localization of R, at the prime ideal p is the ring of rational numbers with odd denominators and is denoted by $R_{(2)}$.
2. If $p = (0)$ given that R is an integral domain, the localization of R at the prime ideal p is the set of all non-zero divisors of R, denoted by $R_{(0)}$ and its called the quotient field of R which is indeed a field.
3. Let R be a ring and R[x] be the ring of polynomials in x indeterminates, then R_p is the ring of all rational functions, $\frac{h}{j}$ where p is a prime ideal in R and $j \notin p$.
4. Let $R = Z$, p, a prime ideal then

$R_p = \{\frac{a}{b}, (b, p) = 1, a \in R\}$.

5. Let p be an ideal of R for $R - p$ to be a multiplicative subset of R it is necessary and sufficient that p be prime.

Significance of localization

1. It serves as a unifying idea in commutative ring theory.

2. It preserves exactness-by this localization plays an important role in homological algebra which plays a very important role in modern development.

3. Localization preserves the Noetherian property in commutative rings.

4. It is the generalization of the process in the construction of new rings (ring of fractions)

5. The technique is applied in proving some important results about unique factorizations.

REFERENCES

Atiyah, M. F. & Macdonald, I. G. 1969. *Introduction to commutative algebra.* Addison Wesely Publishing company Inco.

Cohn, P. M. 1977. *Algebra* Vol. 2, Bedford Collage University of London, John Willey and sons, Ltd.

Jacob, B. 1969. *Topics in ring theory*, New York

Kuku A. O. 1980. *Abstract Algebra,* Ibadan University Press