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STABLE CONTROL FOR CONTAMINATED AQUIFER IN AN UNCERTAIN ENVIRONMENT

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ABSTRACT
This paper considered contamination of aquifer resulting from petroleum spillage, which is a common phenomenal in the Niger Delta area of Nigeria. We used the model given by Bestman (1987) and assumed that some endogenous variables are built into the system. To achieve a level of desirable state, we expect the state of water and that of petroleum to be as close as possible within some $\varepsilon -$ neighbourhood. A method to compute a stable control $\bar{u}$ (.) that will minimise the cost of achieving this desirable state was considered in this paper. We apply the result to a second order system.

Keywords: Stable Control, Aquifer, Petroleum Spillage.

INTRODUCTION

Purification of a river containing some element of pollution occurs by oxidation. The most widely accepted measure of degree of pollution in a river is its dissolved oxygen content (DO). If there is an effluent discharge of waste materials, from a municipal or industrial source in the river, it creates a biochemical oxygen demand (BOD). In many rivers, the oxygen level is reduced below that needed for the survival of fishes and other aquatic life.

Method for raising the DO levels to a specific value include (Chen & Lee, 1987)

(i) Enhance in-plant abatement.

(ii) Flow augmentation from a reservoir system and

(iii) Artificially induced in-stream aeration.

The last method represents a logical way to restore oxygen in a simulated ecological manner. It is by far the cheapest method (Lee & Leitmann, 1985). Pollution of water occurs most frequently at the lower Niger Delta of Nigeria, as a result of petroleum spillage due to exploration activities. Artificially induced in-stream aeration method could have being the best method to restore polluted water to drinkable level in this region; however this is non-existence due to some environmental factors. The best considered, option available is to breed micro-organism that can consume these pollutants.

To implement this method on a large scale represents significant financial investments which require that our system must be designed optimally. In this paper we considered pollution due to petroleum spillage, and we use the model given by Bestman (1987). We also assumed that some endogenous variables are built into the system which we refer to as uncertainty. In order to reach some level of desirable state, we expect the state of water and petroleum to be as closed as possible within some tolerant level. We set out in this paper to compute a stable control $\bar{u}(.) \in U$ that will minimise the cost of achieving this desirable state.

The Model: Denote the water and crude oil phase respectively by W and P then using the model due to Bestman (1987), the governing equations becomes

$$\begin{cases} \dfrac{\sigma\partial W}{\partial t} = \dfrac{\partial^2 W(t)}{\partial x^2} + \dfrac{\partial^2 W(t)}{\partial y^2} \\ k\sigma \dfrac{\partial P}{\partial t} = \dfrac{\partial^2 P(t)}{\partial x^2} + \dfrac{\partial^2 P(t)}{\partial y^2} \end{cases} \quad ...(1)$$

In equation (1), the equilibrium state is achieved where the L.H.S. is identically zero, and the solution at this point is an harmonic expression of the form

$$\nabla^2 W = 0 \quad \text{and} \quad \nabla^2 P = 0 \quad ...(2)$$

In real life situation, water phase is subject to uncertain disturbance such as contamination resulting from industrial waste. This contamination increases the growth of micro-organism that consumes the available oxygen in water. If this situation continues unchecked, many aquatic lives would die. We shall call these organic waste, disturbances.

Let $v(t)$ denote the disturbances at time t and assume that it is bounded, hence there exist $M > 0$ such that

$$\|v(t)\| \le M \quad \text{and} \quad v(t) \in V \subset R^n \quad ...(3)$$

Since uncertainty affects the water phase equation. (1) becomes

$$\begin{cases} \sigma \dfrac{\partial W}{\partial t} = \dfrac{\partial^2 W(t)}{\partial x^2} + \dfrac{\partial^2 W(t)}{\partial y^2} + v(t) \\ k\sigma \dfrac{\partial P}{\partial t} = \dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P(t)}{\partial y^2} \end{cases} \quad \text{...(4)}$$

In order to reduce the level of crude oil pollution and also to drive the water quality to a desirable steady state in the presence of uncertainty and to maintain it there, we introduce two controls $u^{(1)}(t)$ and $u^{(2)}(t)$. The system equation (4) becomes

$$\begin{cases} \sigma \dfrac{\partial W}{\partial t} = \dfrac{\partial^2 W(t)}{\partial x^2} + \dfrac{\partial^2 W(t)}{\partial y^2} + v(t) + u^{(1)}(t) \\ k\sigma \dfrac{\partial P}{\partial t} = \dfrac{\partial^2 P(t)}{\partial x^2} + \dfrac{\partial^2 P(t)}{\partial y^2} + u^{(2)}(t) \end{cases} \quad \text{...(5)}$$

The initial and boundary conditions are

$$\begin{cases} W(x,0) = P(x,0) = 0 \\ W(x,1) = P(x,1) = G \end{cases} \quad \text{...(6)}$$

$$\begin{cases} W(L,y) = H_1 \\ P(0,y) = H_2, L = l/b \end{cases}$$

Where

$$\begin{aligned} &\sigma = s/n \\ &s = specific, storage \\ &n = porosity \\ &l = length, of, acquifer \\ &b = breadth \\ &k = permeability \end{aligned} \quad \text{...(7)}$$

Solution Methodology: We assume that the system (5) satisfies a trial solution of the form defined by

$$\{W(x,y,t) = \sum_{j=1}^{\infty} \alpha_j^{(1)}(t) \sin j\pi x \sin j\pi y$$

$$\{P(x,y,t) = \sum_{j=1}^{\infty} \alpha_j^{(2)}(t) \sin j\pi x \sin j\pi y \quad \text{...(8)}$$

$$\{\ u^{(i)}(t) = \sum_{j=1}^{\infty} u_j^{(i)}(t) \sin j\pi x \sin j\pi y$$

$$\{v(t) = \sum_{j=1}^{\infty} v_j(t) \sin j\pi x \sin j\pi y$$

Now,

$$\begin{cases} W_t(x,y,t) = \sum_{j=1}^{\infty} \dot{\alpha}_j^{(1)}(t) \sin j\pi x \sin j\pi y \\ W_{xx}(x,y,t) = \sum_{j=1}^{\infty} \alpha_j^{(1)}(t)(-\pi^2 j^2) \sin j\pi x \sin_j \pi y \\ W_{yy}(x,y,t) = \sum_{j=1}^{\infty} \alpha_j^{(1)}(t)(-\pi^2 j^2) \sin j\pi x \sin j\pi y \end{cases} \quad \text{...(9)}$$

Similarly,

$$\begin{cases} P_t(x,y,t) = \sum_{j=1}^{\infty} \dot{\alpha}_j^{(2)}(t) \sin j\pi x \sin j\pi y \\ P_{xx}(x,y,t) = \sum_{j=1}^{\infty} \alpha_j^{(2)}(t)(-\pi^2 j^2) \sin j\pi x \sin j\pi y \\ P_{yy}(x,y,t) = \sum \alpha_j^{(2)}(t)(-\pi^2 j^2) \sin j\pi x \sin j\pi y \end{cases} \quad \text{...(10)}$$

On the basis of equations (9) and (10) we re-write (5) as follows:

$$\begin{cases} \sigma\dot{\alpha}_j^{(1)}(t) = -2j^2\pi^2\alpha_j^{(1)}(t) + u_j^{(1)}(t) + v_j(t) \\ k\sigma\dot{\alpha}_j^{(2)}(t) = -2j^2\pi^2\alpha_j^{(2)}(t) + u_j^{(2)}(t) \end{cases} \quad \text{...(11)}$$

Re-arranging (11) we have:

$$\begin{cases} \dot{\alpha}_j^{(1)}(t) = -2j^2 \dfrac{\pi^2}{\sigma} \alpha_j^{(1)}(t) + \dfrac{u^{(1)}(t)}{\sigma} + \dfrac{v_j(t)}{\sigma} \\ \dot{\alpha}_j^{(2)}(t) = -2j^2 \dfrac{\pi^2}{k\sigma} \alpha_j^{(2)}(t) + \dfrac{1}{k\sigma} u_j^{(2)}(t) \end{cases} \quad \text{...(12)}$$

Assume permeability rate is unity then we have

$$\begin{cases} \dot{\alpha}_j^{(1)}(t) = -2j^2 \dfrac{\pi^2}{\sigma} \alpha_j^{(1)}(t) + \dfrac{1}{\sigma} u_j^{(1)}(t) + \dfrac{1}{\sigma} v_j(t) \\ \dot{\alpha}_j^{(2)}(t) = -2j^2 \dfrac{\pi^2}{\sigma} \alpha_j^{(2)}(t) + \dfrac{1}{\sigma} u_j^{(2)}(t) \end{cases} \quad \text{...(13)}$$

Consider the following transformation

$$\begin{cases} z_j(t) = (\alpha_j^{(1)}(t) - \alpha_j^{(2)}(t)) \\ u_j(t) = (u_j^{(1)}(t) - u_j^{(2)}(t)) \end{cases} \quad \text{...(14)}$$

Where $u_j(t)$ is chosen to reduce petroleum pollution to a tolerable level such that the state of the water and petroleum could be as closed as possible within some ε -neighbourhood.

Equation (13) becomes

$$\dot{z}_j(t) = -2j\frac{\pi^2}{\sigma}z_j(t) + \frac{1}{\sigma}u_j(t) + \frac{1}{\sigma}v_j(t); j \in \{1,2,\ldots,N\} \quad \ldots(15)$$

Where N is large enough, according to the reach of the aquifer.

Define

$$\begin{cases} \dot{z}(t) = (\dot{z}_1(t),\ldots,\dot{z}_N(t))^T \\ z(t) = (z_1(t),\ldots,z_N(t))^T \\ u(t) = (u_1(t),\ldots.,u_N(t))^T \\ v(t) = (v_1(t),\ldots.,v_N(t))^T \end{cases} \quad \ldots(16)$$

We put the system of equation (15) as follows:

$$\dot{z}(t) = Az(t) + Bu(t) + Bv(t) \quad \ldots(17)$$

Where $u(t) \in U \subseteq R^N$ and the matrix operators A and B will be defined shortly. Meanwhile we associate a cost functional defined by:

$$Min(z,u) = \int \{z_1^2(t) + \ldots.. + z_N^2(t) + u_1^2(t) + \ldots.. + u_N^2(t)\}dt \quad \ldots(18)$$

Putting (18) more abstractly, we have

$$MinJ(z,u) = \int_0^T \{< z, Qz > + < u, \tilde{R}u >\}dt$$

Subject to

$$\dot{z}(t) = Az(t) + Bu(t) + Bv(t) \quad \ldots(19)$$

Where

$v(t) \in V \subset R^N, u(t) \in U \subset R^N, z(t) \in Z \subset R^N$, a
$\|v(t)\| \le M_1, \|u(t)\| \le M_2$

$< .,. >$ denotes inner product defined on Euclidean space R^N and matrices $Q, \tilde{R}, A, and, B$ are defined as follows:

$$Q = \tilde{R} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & 1 & . & . & . & . \\ . & . & 0 & . & . & . & . \\ . & . & 0 & . & . & . & 0 \\ . & . & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix}_{N\times N}$$

$$A = \begin{bmatrix} -2\frac{\pi^2}{\sigma} & & & & & & \\ . & -8\frac{\pi^2}{\sigma} & & & & & \\ .. & & & & & & \\ . & & . & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & & -2\frac{N^2\pi^2}{\sigma} \end{bmatrix}_{N\times N}$$

$$B = \begin{bmatrix} \frac{1}{\sigma} & & & & & & \\ & \frac{1}{\sigma} & & & & & \\ & & . & & & & \\ & & & .. & & & \\ & & & & . & & \\ & & & & & . & \\ & & & & & & \frac{1}{\sigma} \end{bmatrix}_{N\times N}$$

We now define the Hamiltonian for (19) as follows;

$$H(z,u,v,\eta) = z(t)^T Qz(t) + u(t)^T \tilde{R}u(t) + \eta^T(Az(t) + Bu(t) + Bv(t)) \quad \ldots(20)$$

The adjoint equation for (20) satisfy

$$\begin{cases} \dot{\eta}^T = -\eta^T A - 2z(t)Q \\ \dot{z}(t) = Az(t) + Bu(t) + Bv(t) \\ 0 = 2u(t)^T\tilde{R} + +B^T\eta \end{cases} \quad \ldots(21)$$

The transversality condition takes the form $\eta(T) = 0$

From (21) $\bar{u}(t) = -\frac{1}{2}\tilde{R}^{-1}B^T\eta \quad \ldots(22)$

Let $\eta(t) = 2K(t)z(t) \quad \ldots(23)$

Where K (t) is $n \times n$ symmetric matrix, and from (23)

$$\dot{\eta}(t) = 2K(t)z(t) + 2K(t)\dot{z}(t)$$

And from (21)

$$\dot{\eta}(t) + A^T\eta + 2z(t)Q = 0 \quad \ldots(24)$$

Therefore

$$2\dot{K}(t)z(t) + 2K\dot{z}(t) + 2A^T Kz(t) + 2z(t)Q = 0 \quad ...(25)$$

From (25) it implies that

$$\dot{K}(t)z(t) + K\dot{z}(t) + A^T Kz(t) + z(t)Q = 0 \quad ...(26)$$

In Abiola & Solarin (2009), we assumed that $\bar{u}(t)$ and $\bar{v}(t)$ are optimal strategies for the minimiser and maximiser such that these are denoted by

$$\begin{cases} \bar{u}(t) = \bar{\gamma}(\bar{z}(t)) \\ \bar{v}(t) = \bar{\alpha}(\bar{z}(t)) \end{cases} \quad ...(27)$$

Where

$\bar{z}(t) : [0.T] \to R^N$ is a solution of the equation described in (19).Therefore from (22) and (23)

$$\bar{u}(t) = -\tilde{R}^{-1} B^T K\bar{z}(t) \quad ...(28)$$

If the worst situation is assumed for the controller, then we set

$$\bar{v}(t) = -m\bar{u}(t) \quad ...(29)$$

and K is computed from

$$\dot{K}A + A^T K - \beta KSK + Q = 0 \quad ...(30)$$

Where
$S = B\tilde{R}^{-1}B^T$ and $\beta = (1-m)$.

Numerical Example: We consider a second order system and for our simulation purpose we let $\sigma = 2,$ then the resulting problem is

$$MinJ(z,u) = \int_0^T \{z_1^2(t) + z_2^2(t) + u_1^2(t) + u_2^2(t)\}dt$$

Subject to

$$\dot{z}_1(t) = -\pi^2 z_1(t) + 0.5u_1(t) + 0.5v_1(t)$$

The problem is put into matrix form as follows:

$$MinJ(z,u) =$$

$$\int_0^T \{(z_1, z_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + (u_1, u_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\}dt$$

Subject to

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -4\pi^2 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

We applied the algorithm described by Abiola & Solarin (2009) to compute a stable control displayed in Fig. 1.

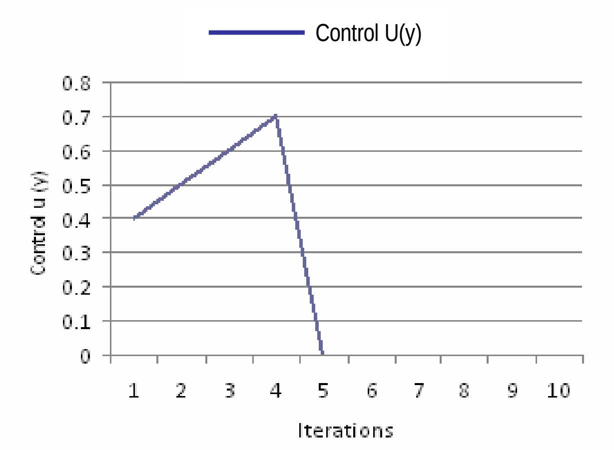

FIG. 1. STABLE CONTROL

The control displayed in Fig. 1, stabilises at iterations 5. The implication of this is that the state of water and that of petroleum become as close as possible as from this point. In practical term at a certain reach of the aquifer, which is achievable, the state of water and petroleum become very close thus reducing the cost of getting to this desirable state of event.

Conclusion: In this paper we have applied the result of Abiola & Solarin (2009) in computing a stable control $\bar{u}(t)$ in the management of water pollution resulting from petroleum spillage. This was based on the model earlier developed by Bestman (1987) which we modified to accommodate some uncertainties. The result in this paper has clearly demonstrated that solution problem resulting from petroleum spillage could be solved, and it only require optimal selection of strategy for achieve this lofty objective.

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