Short Communication Report

ON POWERS OF MULTISETS

*TELLA, Y. & DANIEL, S.

Department of Mathematical Sciences, Kaduna State University, Kaduna, Nigeria.
*yot2002ng@yahoo.com

ABSTRACT
A multiset is a collection of objects in which repetition of elements is significant. In this paper an attempt is made to extend powers of real numbers to powers of multisets.

Key words: Multiset, Powers of a Multiset

INTRODUCTION
The notion of Multiset is a concept appearing in many areas of mathematics, computer science and managerial science. Intuitively, multisets are a generalization of sets in which elements can occur more than once. The number of occurrences of an element is called its *multiplicity*. The multiset $\{a,a,b\}$, which is not a set, is distinct from $\{a,b\}$; the multiplicity of a is 2 in the former and 1 in the later. The word *multiset* has been coined by De Bruijin in a private communication with Knuth(1981). Other terms have appeared here and there in the literature such as bag, heap, occurrence set, fireset, e.t.c. A survey of the theory of multisets can be found in Blizard (1991).

Even if the concept of multiset is present in mathematics, logic and, more and more, in computer science, it has long been eclipsed by the classical Cantorian view of a set. Cantor (yr) states that every 1 is a different instance of 1 and may be distinguished from other 1's. However this looks a bit artificial. We confront a number of situations in life when we have to deal with collections of elements in which duplicates are significant.

Thus from a practical point of view, multisets are very useful structures as they arise in many areas of mathematics. The prime factorization of integers *n*>0 is a multiset whose elements are primes. Every monic polynomial *f*(*x*) over the complex numbers corresponds in a natural way to the multiset of its roots. Other examples of multisets include the zeros and poles of meromorphic functions, invariants of matrices in a canonical form etc. Coming back to computer science, the article of Dershowitz & Manna (1979) introduced a multiset ordering and used it to prove program termination. Actually, given a well-founded ordering on elements of the multiset, it is possible to derive a well-founded ordering on multisets themselves that allows elegant proofs of termination which otherwise could be awkward.

Research on the multiset theory has not yet gained ground and is still in its infant stages. The research carried out so far shows a strong analogy in the behavior of multisets and sets and it is possible to extend some of the main notions and results of sets to that of multisets. Since the case in consideration has a wider generality (compared to that of sets), the results obtained for multisets are technically more complicated. This paper is an attempt to explore the theoretical aspects of multisets by extending the notion of powers of real numbers to the multiset context.

PRELIMINARIES AND BASIC DEFINITIONS
Definition 1. A collection of elements containing duplicates is called a multiset. Formally, a multiset A over a set S is a cardinal-valued function. That is, A on S is a map from S to the set N of natural numbers denoted $A: S \to \mathrm{N}$. Accordingly, multisets are also denoted by the commonly used function symbols $f, g, h, \ldots$

For objects $x \in A$, $A(x)$ or $M_A(x)$ or $f(x)$ denotes the multiplicity of x in A or f and the sum of all the multiplicities of A is the cardinality of A denoted $|A|$.

It follows by definition that $A(x) > 0 \quad \forall x \in A$. A class of all finite multisets over S can be represented by:

$$M(S) = \{A | A : S \to N \text{ and } A(x) = 0 \text{ for all but finitely many } x \in S\}$$

Here, S is called the generating or ground or base set for $M(S)$.

A multiset can also be represented as the set of pairs. For example,

Let $S = \{x_1, x_2, \ldots\}$, then
$A = \{\langle M_A(x_1), x_1\rangle, \langle M_A(x_2), x_2\rangle, \ldots\}$ or
Or $A = \{M_A(x_1) \circ x_1, M_A(x_2) \circ x_2, \ldots\}$
or $A = \{A(x_1) \circ x_1, A(x_2) \circ x_2, \ldots\}$ (for short) is a multiset.

Definition 2. Every multiset is associated with a particular set, namely the *root* set containing exactly the generators or objects of that multiset. In particular, let M be a multiset over the generic set S, then its root set denoted M^* is defined:

$$M^* = \{a | a \in S \wedge M(a) > 0\}$$

For any multiset M, we define the predicate $Set(M)$ by:
$Set(M): M = \phi \vee \forall x \forall n(M(x) = n \rightarrow n = 1)$

Definition 3. Two multisets A and B over the generic set S are *equal* or the *same*, written as $A = B$ if for any element $x \in S$, $M_A(x) = M_B(x)$ or $A(x) = B(x)$.

Definition 4. The following operations with multisets are defined: Arithmetic addition

$M + N = \{(M+N)(x) \circ x | (M+N)(x) = M(x) + N(x)\}$

Multiplication by a scalar $k.M = \{(kM)(x) \circ x | (kM)(x) = kM(x),\ k \in N\}$

Arithmetic multiplication $M.N = \{(M.N)(x) \circ x | (M.N)(x) = M(x)N(x)\}$

Raising to arithmetic power $A^n = \{A^n(x) \circ x | A^n(x) = (A(x))^n \quad n \in N\}$ (see Blizard, 1989 and Petrovsky, 2004).

POWERS OF MULTISETS

Proposition 1. Let $M, N \in M(S)$.

(i) ϕ^0 is undefined
(ii) $M^0 = M^*(M \neq \phi)$ and $M^1 = M$
(iii) $M^*.M = M.M^* = M$
(iv) $(M.N)^n = M^n.N^n$
(v) $M^m.M^n = M^{n+m}$
(vi) $(M^m)^n = (M^n)^m = M^{mn}$
(vii) $M^n = M^k \rightarrow n = k$
(viii) if $M^* = N^k$ then $M^* = N \wedge k = 0$

Proof:

(i) $\phi^0(x) = (\phi(x))^0 = 0^0$ (which is undefined). Thus ϕ^0 is undefined.

(ii) Since $M \neq \phi$, let $x \in M$, we have $M(x) > 0$ and $x \in M^*$
$M^0(x) = (M(x))^0 = 1 > 0$, thus $x \in M^* \rightarrow x \in M^0$ and

$$M^* \subseteq M^0 \qquad \text{... (1)}$$

But for all $y \in M^0$ we have $M^0(y) = (M(y))^0 = 1$
Thus, $Set(M^0)$ and $(M^0)^* = M^0$. We show that $M^0 \subseteq M^*$
Now $y \in M^0 \rightarrow M^0(y) = (M(y))^0 = 1 > 0$
Thus, $M(y) > 0$ and $y \in M$. In particular $y \in M^*$

Hence, $M^0 \subseteq M^*$ ... (2)

Since $Set(M^0)$ and $Set(M^*)$, we have $M^* = M^0$ (from (1) and (2))

$M^1(x) = (M(x))^1 = M(x)$. Thus $M^1 = M$

(iii) $(M.M^*)(x) = M(x)M^*(x) = 1.M(x) = M(x)$ (since $Set(M^*)$) and
$(M^*.M)(x) = M^*(x)M(x) = 1.M(x) = M(x)$. Therefore, $M^*.M = M.M^* = M$

(iv) $(M.N)^n(x) = ((M.N)(x))^n = (M(x).N(x))^n = (M(x))^n(N(x))^n = M^n(x).N^n(x) = (M^n.N^n)(x)$

Thus we have $(M.N)^n = M^n.N^n$

(v) $(M^n.M^r)(x) = M^n(x)M^r(x) = (M(x))^n(M(x))^r = (M(x))^{n+r} = M^{n+r}(x)$

Therefore $M^n.M^r = M^{n+r}$ and

(vi) $(M^n)^m(x) = (M^n(x))^m = ((M(x))^n)^m = (M(x))^{nm} = (M(x))^{mn} = M^{mn}(x)$ ... (3)

But $((M(x))^n)^m = ((M(x))^m)^n$.

Thus $(M^n)^m(x) = ((M(x))^n)^m = ((M(x))^m)^n = (M^m)^n(x)$
In particular $(M^n)^m = (M^m)^n$ ... (4)

Hence, $(M^n)^m = (M^m)^n = M^{mn}$ (from (3) and (4)).

(vii) Let $M^n = M^k$

By definition, $M^n(x) = M^k(x) \quad \forall x$ i.e. $(M(x))^n = (M(x))^k$ and $n = k$.

(viii) Supposing $M^* = N^k$. Then $M^*(x) = N^k(x) = (N(x))^k \quad \forall x$

In particular, $M^*(x) = 1 = (N(x))^k$ (by definition) and $k = 0$

However if $M^* \neq N$ then , and $M^*(x) \neq (N(x))^k$ for some x. In particular, $M^*(x) \neq N^k(x)$. i. e. $M^* \neq N^k$ (a contradiction).

Thus, *powers* of a multiset M can be defined:

$$M^{k+1} = M^k.M,\ M^1 = M \text{ and } M^0 = M^*$$

It is interesting to note that every polynomial

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ can be written as a polynomial in a multiset M . Thus for the above polynomial $f(x)$, we have $f(M) = a_n M^n + a_{n-1}M^{n-1} + \ldots + a_1 M + a_0 M^0$ a corresponding polynomial multiset.

Now let $M, N \in M(S)$. Then $\log_N^M$ is defined: $\log_N^M = k \leftrightarrow M = N^k$

Proposition 2: Let $M, N, P \in M(S)$ such that $M, N, P \neq \phi$

(i) $\log_P^{(M.N)} = \log_P^M + \log_P^N$

(ii) $\log_P^{M^k} = k\log_P^M$

(iii) $\log_P^{M^*} = 0$

(iv) $\log_N^M = \dfrac{\log_P^M}{\log_P^N}$

Proof:

(i) Let $\log_P^M = k$ and $\log_P^N = r$. By definition, $M = P^k$ and $N = P^r$

Thus, $M.N = P^k P^r = P^{k+r}$ (from proposition 1 (v))
Hence, $\log_P^{(M.N)} = k + r$ (by definition) and $\log_P^{(M.N)} = \log_P^M + \log_P^N$.

(ii) Let $\log_P^M = r$ By definition, $M = P^r$. Thus, $M^k = (P^r)^k = P^{kr}$ and $\log_P^{M^k} = kr$ (by definition). Hence, $\log_P^{M^k} = k \log_P^M$

(iii) Let $\log_P^{M^*} = k$ Then $M^* = P^k$ (by definition).
But $M^* = P^k \rightarrow k = 0$ (Proposition 1 (viii)). Thus $\log_P^{M^*} = 0$

(iv) Let $\log_N^M = k$. We have $M = N^k$ (by definition)

Taking logarithm on both sides to base P, we have
$\log_P^M = \log_p^{N^k} = k \log_P^N$ (from (ii) above). Hence, $k = \dfrac{\log_P^M}{\log_P^N}$ i. e. $\log_N^M = \dfrac{\log_P^M}{\log_P^N}$

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